ECE 301
Transients

Second Order Systems

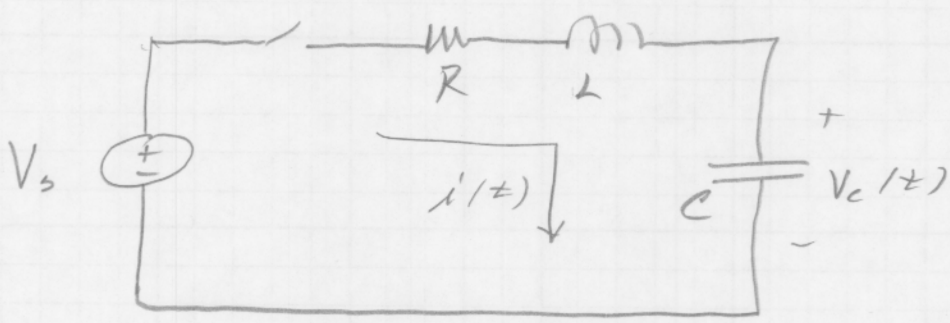
Circuits with series or parallel RLC result in a 2nd order differential equation of the form

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + b\,x(t) = f(t)$$

Since we are assuming a constant source, $f(t)$ = constant.

To better illustrate this consider the

Series RLC



We can write

$$R\,i(t) + L\frac{di}{dt} + V_c(t) = V_s \qquad (1)$$

The next step depends on whether we want a solution for $V_c(t)$, or $i(t)$. Let us go for $V_c(t)$ first.

In Eq (1) we use

$$i = C\frac{dV_c}{dt}$$

to obtain

$$RC\frac{dV_c}{dt} + LC\frac{d^2V_c}{dt^2} + V_c(t) = V_s$$

or

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c(t)}{LC} = \frac{V_s}{LC} \qquad (2)$$

If we want a solution in $i(t)$ we can use

$$R\,i(t) + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt + V_c(0) = V_s \qquad (3)$$

Taking the derivative wrt t gives

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i(t)}{C} = 0$$

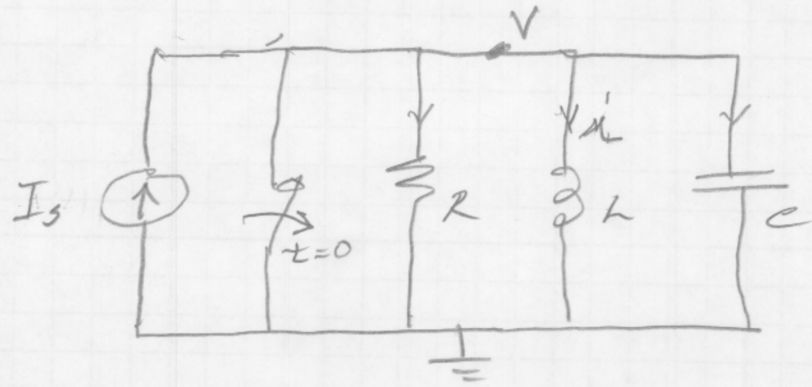
or

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i(t)}{RC} = 0 \qquad (4)$$

Notice that characteristic equation is the same for both (2) and (4) and this is expected. The C.E. is

$$\boxed{s^2 + \frac{R}{L}s + \frac{1}{RC} = 0} \qquad (5)$$

Now consider the parallel RLC



Using nodal equations,

$$\frac{V}{R} + C\frac{dV}{dt} + i = I_s \qquad (6)$$

If we want a solution in $i_L(t)$, we use

$$v(t) = L\frac{di}{dt}$$

and substitute into Eq. (6):

$$\frac{L}{R}\frac{di}{dt} + LC\frac{d^2i}{dt^2} + i(t) = I_s$$

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i(t)}{LC} = \frac{I_s}{LC} \qquad (7)$$

The C.E. is

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \qquad (8)$$

If we want a solution in $v(t)$ we use

$$i = \frac{1}{L}\int_0^t v(t)\,dt + i(0) \qquad (9)$$

substitute Eq (9) into Eq (6)

$$\frac{V}{R} + C\frac{dV}{dt} + \frac{1}{L}\int V(t)\,dt + i(0) = I_s$$

Take the derivative of the above wrt t to find

$$\frac{1}{R}\frac{dV}{dt} + C\frac{d^2V}{dt^2} + \frac{V(t)}{L} = 0$$

and then

$$\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{V(t)}{LC} = 0 \qquad (10)$$

The characteristic equation is

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \qquad (11)$$

Note that Eq (11) and Eq (8) are the same.

In the final analysis we need to solve a diff. eq. of the form

$$\frac{d^2x}{dt^2} + A\frac{dx}{dt} + Bx(t) = C \qquad (12)$$

or

$$\frac{d^2x}{dt^2} + D\frac{dx}{dt} + Ex(t) = 0 \qquad (13)$$

As the most general case let us use Eq (12). Also, for help in explaining things let us use a special form of

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x(t) = C \qquad (14)$$

where

$$2\zeta\omega_n = A$$

$$\omega_n^2 = B$$

$\underline{\omega_n}$: is called the undamped natural resonant frequency

ζ : is called the damping coefficient.

The general solution of (14) is

$$x(t) = x_p(t) + x_c(t)$$

where

$$x_p(t) = \frac{C}{B} \qquad (15)$$

(refer back to Eq (12).

$x_p(t)$ is the particular solution.

The complimentary solution will be of the form

$$x_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where s_1 and s_2 are the roots of the characteristic equation. The characteristic equation is

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

From the quadratic equation we know

$$s_1 = \frac{-2\xi\omega_n + \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2}$$

$$s_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1} \qquad (16)$$

and

$$s_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1} \qquad (17)$$

There are 3 cases to consider for these roots:

<u>Case 1</u>: $\xi > 1$

This is the case of overdamping. We have two real and distinct roots; Our solution is

$$x(t) = \frac{C}{B} + k_1 e^{s_1 t} + k_2 e^{s_2 t} \qquad (18)$$

We will need to know $x(0)$ and $\frac{dx(0)}{dt}$ in order to solve for k_1 and k_2.

CASE 2 ; $\xi = 1$

This is called critical damping. For this case

$$s_1 = s_2 = -\xi\omega_n$$

AND

$$x(t) = \frac{c}{k} + (k_1 + t k_2)e^{s_1 t} \qquad (15)$$

Again, we must know two I.C. to solve for k_1 and k_2. Namely

$$x(0), \quad \frac{dx(0)}{dt}$$

CASE 3 ; $\xi < 1$

This is the case of underdamping

For

$$s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

we write

$$s = -\xi\omega_n \pm j\,\omega_n\sqrt{1-\xi^2}$$

$$s_1 = -\xi\omega_n + j\,\omega_d ; \quad s_2 = -\xi\omega_n - j\,\omega_d$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

ω_d is called the damped natural resonant frequency.

We write

$$x(t) = \frac{c}{B} + k_1 e^{-(\xi\omega_n + j\omega_d)t} + k_2 e^{-(\xi\omega_n - j\omega_d)t}$$

$$= \frac{c}{B} + k_1 e^{-\xi\omega_n t} \cdot e^{-j\omega_d t} + k_2 e^{-\xi\omega_n t} e^{j\omega_d t}$$

$$= \frac{c}{B} + e^{-\xi\omega_n t}\left[k_2 e^{j\omega_d t} + k_1 e^{-j\omega_d t}\right]$$

$$= \frac{c}{B} + e^{-\xi\omega_n t}\left[k_2(\cos\omega_d t + j\sin\omega_d t\right.$$
$$\left. + k_1(\cos\omega_d t - j\sin\omega_d t\right]$$

collecting terms

$$x(t) = \frac{c}{B} + e^{-\xi\omega_n t}\left[(k_1 + k_2)\cos\omega_d t\right.$$
$$\left. + j(k_2 - k_1)\sin\omega_d t\right]$$

or

$$x(t) = \frac{c}{B} + e^{-\xi\omega_n t}\left[A_1\cos\omega_d t + A_2\sin\omega_d t\right]$$

We find A_1 and A_2 using initial conditions;

$x(0)$, $\frac{dx(0)}{dt}$

Next for illustrations