USING OPERATIONAL AMPLIFIERS TO SOLVE DIFFERENTIAL EQUATIONS

ECE 300

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(rough notes)

Another Look At Operational Amplifiers

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Earlier we how op-amps could be use to sum signals. We also learned how to analyze the performance of the op amp with various configurations of resistor and sources connected to the non inverting, inverting inputs, and the output! For the most part we used nodal analysis.

We now consider further use of the op amp when capacitors are used along with resistors. Inductors are seldom, if ever, used with op amps because of their nature to introduce noise. The voltage across the inductor is given by

$$V_L(t) = L\frac{di}{dt}$$

any variation in the current, produced by noise has it's derivative taken which accentuates the noise.

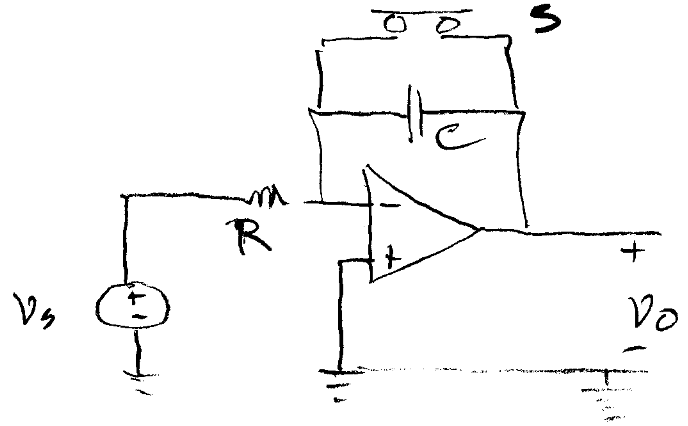

Figure 1: The op amp configured as an integrator.

First, one would seldom use the configuration of Figure 10.1 to integrate a signal unless the switch S is close until just before the input signal is applied. If the switch is not used, noise is integrated prior to t=0. Often the integration of the noise cause the output to reach 12 V (741) which places the op-amp output at saturation. Even if the switch is used, if the resistor R is in the meg ohm range, significant noise will be introduced for $t > 0$ even if the signal v_s is applied.

Nevertheless, we analyze the configuration of Figure 1. Perhaps with the hope that significant noise is not present when we use the configuration, at and alone.

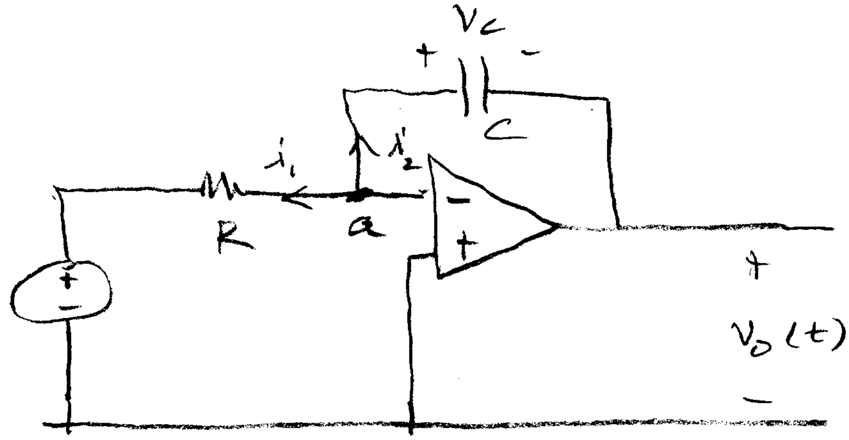

Figure B: Basic circuit for using the op amp as an integrator.

At point "a" we write;

$$-\frac{v_s}{R_1} + C\frac{dv_c}{dt} = 0 \qquad (1)$$

Since point "a" is at zero potential, we have

$$V_o + V_c = 0$$

OR

$$V_c = -V_o \qquad (2)$$

using Equation (2) in Equation (1) give

$$\frac{dV_o(t)}{dt} = -\frac{1}{RC} V_s(t) \qquad (3)$$

OR

$$\int_{V_o(t_o)}^{V(t)} V_o(t) = -\frac{1}{RC} \int_{t_o}^{t} V_s(t)dt$$

OR

$$V_o(t) = -\frac{1}{RC} \int_{t_o}^{t} V_s(t)dt + V_o(t_o) \qquad (4)$$

We usually take $t_o = 0$. Equation (4) becomes

$$V_o(t) = -\frac{1}{RC} \int_{0}^{t} V_s(t)dt + V_o(0) \qquad (5)$$

Equation (5) clearly shows that the circuit integrates the input signal, $V_s(t)$. We note that the circuit does not, as it stands, provide for $V_o(0)$ at $t=0$. There are straightforward ways available for including $V_o(0)$. We will not discuss the issue further at this point.

Solving A First Order Differential Equation

Consider the following expression.

$$\frac{dx(t)}{dt} + a x(t) = s(t)$$

We write this as follows, for convenience.

$$\dot{x}(t) = s(t) - a x(t) \qquad (6)$$

We note that if both sides of Equation (6) is integrated we have,

$$x(t) = \int [s(t) - a x(t)]\, dt$$

Since there is a sign change associated with the input to output signal of a single op-amp, we think in terms that if $\dot{x}(t)$ is applied as the input signal to Figure 3, the output will be $-x(t)$. That is

$$\underset{\text{in}}{\dot{x}(t)} \xrightarrow{\int} -x(t) \qquad (7)$$

OR

$$\underset{\text{inp}}{-\dot{x}(t)} \xrightarrow{\int} x(t) \qquad (8)$$

From Equation (6) we can write

$$-\dot{x}(t) = a x(t) - s(t) \qquad (9)$$

Equation (9) invites us to find a way to apply $a\,x(t) - s(t)$ to the input of the op-amp and the output will be $x(t)$.

How can we do this? Consider the circuit of Figure 4.

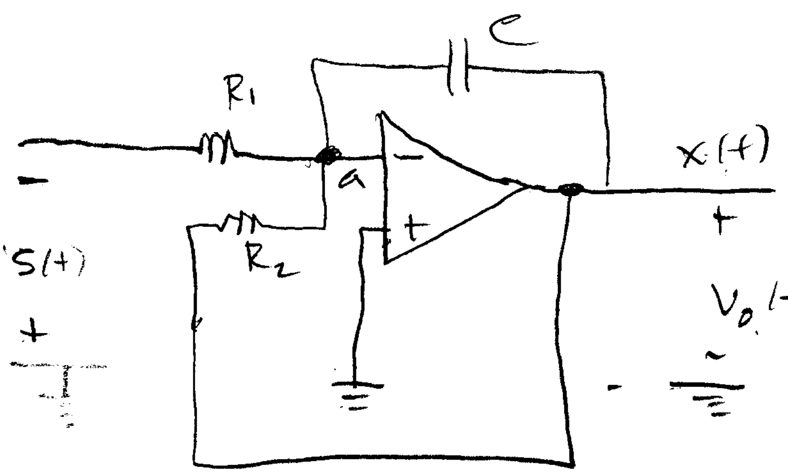

Figure 4: Circuit for solving a first order differential equation.

The procedure is to assume we have the desired $x(t)$ and the configure a circuit that provides the input to give $x(t)$.

It is easy to see that the output of the circuit in Figure 4, in terms of the input, is given by

$$x(t) = \frac{1}{R_1 C}\int s(t) - \frac{1}{R_2 C} x(t) \qquad (10)$$

We also recall from Equation (4)

$$-\dot{x}(t) = -s(t) + a\,x(t) \qquad (11)$$

$$\int -\dot{x}\,dt = \int \left(-s(t) + a\,x(t)\right) dt \longrightarrow x(t) \qquad (12)$$

So all we need to observe is that

$$\left| \int s(t)\,dt \right| = \left| \frac{1}{R_1 C} \int s(t)\,dt \right| \qquad (13)$$

so $$\frac{1}{R_1 C} = 1 \qquad (14)$$

Also $$\left| \int a\,x(t) \right| = \left| \frac{1}{R_2 C} \int x(t) \right| \qquad (15)$$

or $$\frac{1}{R_2 C} = a \qquad (16)$$

Example 11:

Given the differential equation

$$\frac{dx(t)}{dt} + 4x(t) = u(t) \qquad (17)$$

Use the circuit of Figure 4 to solve this equation. Assume $C = 10\mu F$. Find R_1 and R_2.

Solution:

The analytical solution of Equation(16) is;

$$x(t) = (1 - e^{-4t})\,u(t) \qquad (18)$$

which appears as in Figure 5.

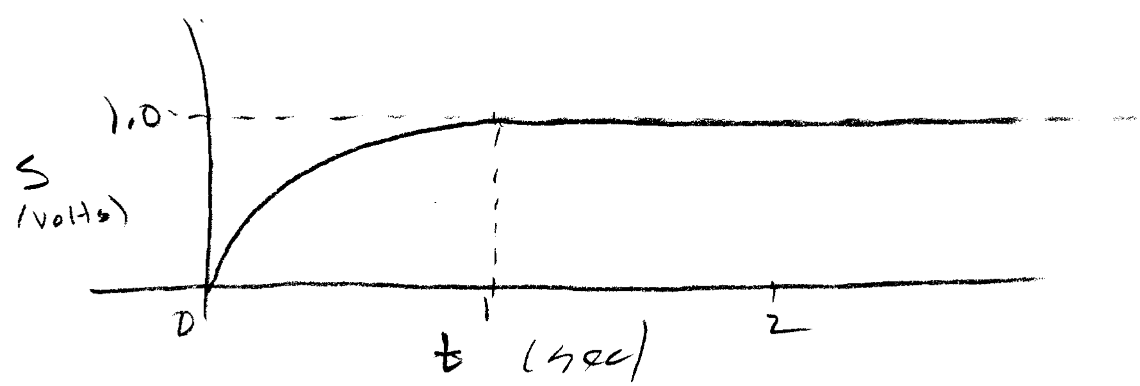

Figure 5: Sketch of the solution for Equation (17).

We recall, that if $s(t) = u(t)$

$$\frac{1}{R_1 C} = 1$$

so

$$R_1 = \frac{1}{C} = 100\,K\Omega \qquad (19)$$

We also recall, Equation (16), that

$$\frac{1}{R_2 C} = 4$$

so,

$$R_2 = 25\,K\Omega \qquad (20)$$

Our circuit would set up as

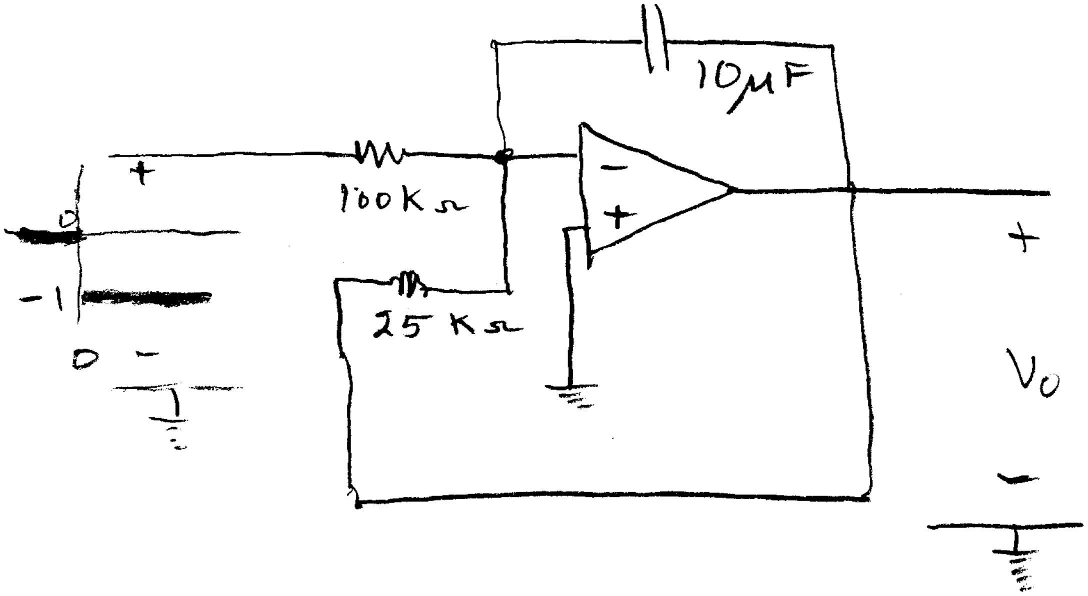

Figure 6: Op-Amp configuration for solving the DE of Equation (17)

2x.4x5

Example 2

Develop an op-amp configuration that solves the following differential equation.

$$\ddot{x}(t) + 4\dot{x}(t) + 25 = u(t) \qquad (21)$$

One notes that this differential equation has complex roots and will give an underdamped response. We compare this to the standard form;

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = u(t) \quad (22)$$

$$\omega_n = 5, \quad \xi = 0.4$$

In a general sense Equation (21) can be written as

$$\ddot{x}(t) + a\dot{x}(t) + b x(t) = s(t) \quad (22)$$

Figure 7 gives an appropriate circuit for solving Equation (21).

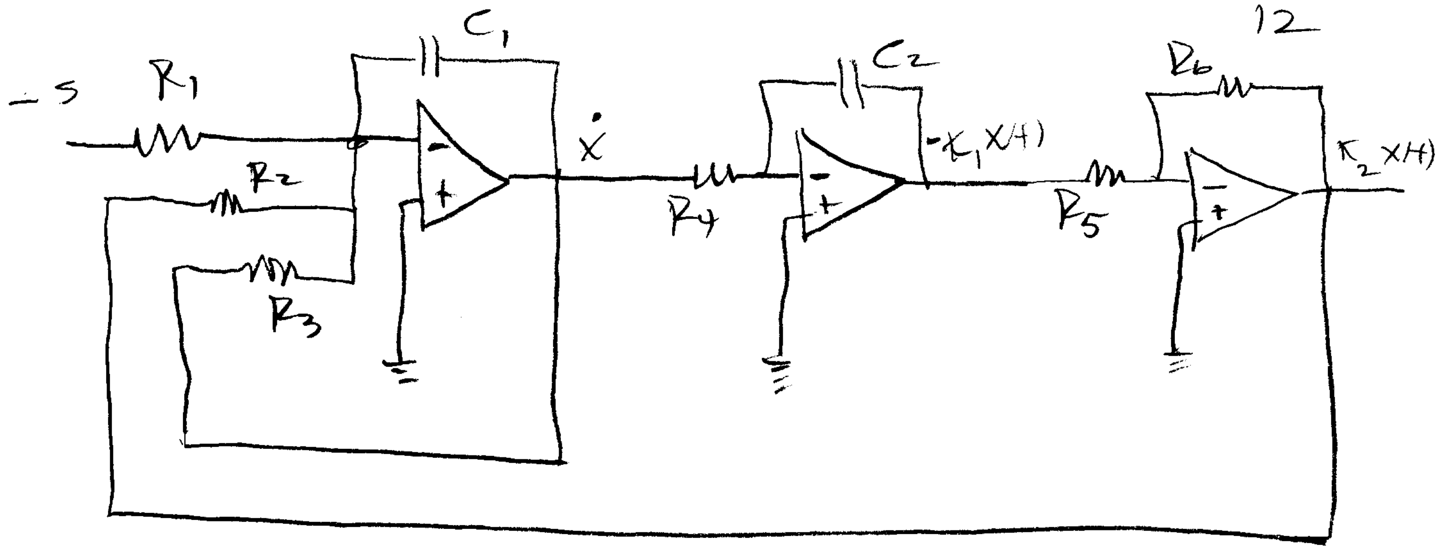

Figure 7: Op-Amp setup for solving a 2nd order differential equation.

We have

for $s(t)$
$$\frac{1}{R_1 C_1} \times \frac{1}{R_4 C_2} \times \frac{R_6}{R_5} = 1 \qquad (23)$$

for $x(t)$
$$\frac{1}{R_2 C_1} \times \frac{1}{R_4 C_2} \times \frac{R_6}{R_5} = b \qquad (24)$$

for $\dot{x}(t)$
$$\frac{1}{R_3 C_1} = a \qquad (25)$$

The procedure is to assume values for C_1, C_2, R_6 & R_5 then solve for R_1, R_2, R_3, R_4 to satisfy Equations (23), (24), (25). We further assume a value of either R_1 or R_2 in this procedure.