1. The top surface of an infinitely large ideal conductor plate is at the plane \( z = 0 \) of the Cartesian coordinate system. The conductor plate is grounded (i.e. at a potential 0). A positive point charge \( Q \) is located at \((0, 0, d/2)\). Assuming free space (i.e. vacuum, with permittivity \( \varepsilon_0 \)), find the following:

\( \text{(a)} \) the electric field and potential at any point with \( z < 0 \).

\( \text{(b)} \) the electric field and potential at any point on the \( z \) axis with \( z > 0 \). (You need to find the expression for the field \( E \) and potential \( V \) in terms of position \( z \)), and

\( \text{(c)} \) the electric field and potential at point \( (0, \sqrt{3}d, d/2) \).

Caution: The field is a vector while the potential is a scalar. You may express a field in the form of \( E = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} \).

\( \text{(a)} \) \( E = 0 \), \( V = 0 \) for \( z < 0 \).

\( \text{(b)} \) Use the image method.

The image is \(-Q\) at \((0, 0, -d/2)\).

For any point on the \( z \) axis, the field \( E(0, 0, z) = E_z \hat{z} \).

\[ E_z = \frac{Q}{4\pi \varepsilon_0 (3 - \frac{d}{2})^2} - \frac{Q}{4\pi \varepsilon_0 (3 + \frac{d}{2})^2} = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{1}{(3 - \frac{d}{2})^2} - \frac{1}{(3 + \frac{d}{2})^2} \right] \]

and the potential

\[ V(0, 0, 3) = \frac{1}{4\pi \varepsilon_0} \left( \frac{1}{3 - \frac{d}{2}} - \frac{1}{3 + \frac{d}{2}} \right) \]

\( \text{(c)} \) The field due to \(+Q\) is \( E_1 = E_y \hat{y} \), where

\[ E_1 = \frac{Q}{4\pi \varepsilon_0 (\sqrt{3}d)^2} = \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{3d^2} \]

The field due to \(-Q\) is \( E_2 = E_{2y} \hat{y} + E_{2z} \hat{z} \)

\[ E_2 = |E_z| = \frac{Q}{4\pi \varepsilon_0 (2d)^2} = \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{2d^2} \]

\[ E_{2y} = -E_2 \cos 30^\circ = \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{2d^2} \cdot \frac{\sqrt{3}}{2} \]

\[ E_{2z} = E_2 \sin 30^\circ = \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{2d^2} \cdot \frac{1}{2} \]

\[ \vec{E} = E_1 + E_2 = \frac{Q}{4\pi \varepsilon_0} \left[ \left( \frac{1}{3d^2} - \frac{\sqrt{3}}{8d^2} \right) \hat{y} - \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{8d^2} \hat{z} \right] \]

\[ = \frac{Q}{4\pi \varepsilon_0 \cdot d^2} \left[ \left( \frac{1}{3} - \frac{\sqrt{3}}{8} \right) \hat{y} - \frac{1}{8} \hat{z} \right] \]
\[ V = \frac{1}{4\pi \varepsilon_0 (\sqrt{3}d)} - \frac{1}{4\pi \varepsilon_0 (2d)} = \frac{1}{4\pi \varepsilon_0 d} \left( \frac{1}{\sqrt{3}} - \frac{1}{2} \right) \]
2. When we study the parallel plate capacitor, we often assume the field distribution to be the same as if the two plates were infinitely large, yet we use a finite plate area to calculate the capacitance. We live with this inconsistency for convenience: we need to ignore the fringe effect. The approximation is shown in Fig. 1(a), where the electric field lines are straight between the plates, and abruptly stop outside the plates. The real situation is something like Fig. 1(b). Show that the field in Fig 1(a) is unphysical (i.e. it violates some law of physics). Hint: it violates one of the Maxwell equations.

![Diagram of parallel plate capacitor](image-url)

**Fig. 1**

The field distribution of Fig. 1(a) violates the conservativity of electrostatic fields, i.e.

\[ \oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad (\text{i.e. } \nabla \times \mathbf{E} = 0 \text{ in the differential form}) \]

Consider a rectangular loop with sides parallel to the field lines of length \( a \) and the other two sides of arbitrary length as drawn above.

Side AB is within the capacitor, while side CD is out.

If the field were \( \mathbf{E} \) as shown in Fig. 1(a), then

\[ \oint \mathbf{E} \cdot d\mathbf{l} = \int_{A}^{B} \mathbf{E} \cdot d\mathbf{l} + \int_{B}^{C} \mathbf{E} \cdot d\mathbf{l} + \int_{C}^{D} \mathbf{E} \cdot d\mathbf{l} + \int_{D}^{A} \mathbf{E} \cdot d\mathbf{l} \]

\[ = E_a \cdot 0 + 0 + 0 + 0 \]

\[ = E_a \neq 0 \]
3. As shown in Fig. 2, an infinitely large slab of dielectric with a dielectric constant $\varepsilon_r$ is joined at $x = 0$ to another infinitely large slab of dielectric, with the same dielectric constant $\varepsilon_r$. The dielectric slab on the right side ($x > 0$) has a uniform volume charge density $\rho$, while the one on the left side has a charge density $-\rho$. You may take $\rho$ as positive. The thickness of both slabs is $d$. Find the electric field and potential as functions of $x$ for all $x$. If the breakdown field of the dielectric is $E_b$, what is the maximum charge density $\rho_{\text{max}}$ it can carry?

Note: The permittivity of vacuum is $\varepsilon_0$. You may write the permittivity of the dielectric as $\varepsilon = \varepsilon_r \varepsilon_0$. Again, you don’t need to touch your calculator.

With symmetry (of the infinitely large slabs) considered, the $\vec{E}$ field is parallel to $\hat{x}$, i.e. $\vec{E} = E \hat{x}$.

There are two methods.

Method 1: The graphical method:

Imagine a cylinder as shown, w/ a cross-section area $A$. First, consider the top surface is at $x > d$, and bottom surface at $x < -d$.

The total charge enclosed by this cylinder is

$$-\rho A d + \rho A d = 0$$

There is no other charge in the space, therefore

$$E = 0 \text{ for } x > d \text{ and } x < -d.$$
At the bottom surface, we have $E = 0$. Therefore, according to Gauss's law,

$$\varepsilon \varepsilon E(x) A = -\rho A(d - x)$$

$$\therefore E(x) = -\rho (d - x) / \varepsilon \text{ for } 0 < x < d$$

Similarly, keep the top surface at $x = d$, and move the bottom surface to the left:

$$+\rho A x + \rho A d = \rho A(d + x)$$

is the total enclosed charge.

$$-E(x) A = \rho A(d + x)$$

$$\therefore E(x) = -\rho (d + x) / \varepsilon$$

for $-d < x < 0$

$$|E_{\text{max}}| = \rho d / \varepsilon \text{ at } x = 0$$

For $0 < x < d$.

$$V(x) = V(0) - \int_0^x E(x) \, dx$$

$$= 0 + \frac{\rho}{\varepsilon} \int_0^x (d - x) \, dx$$

$$= \frac{\rho}{2\varepsilon} x(2d - x)$$

[You could do the integral graphically by finding the shaded area]
For \( x > d \), \( E(x) = 0 \)

\[
V(x) = V(d) = \frac{\rho}{2\varepsilon} d^2
\]

For \( -d < x < 0 \),

\[
V(x) = V(0) - \int_0^x E(x) \, dx
\]

\[
= 0 - \frac{\rho}{2\varepsilon} \int_0^x (d+x) \, dx = \frac{\rho}{2\varepsilon} x (2d+x)
\]

For \( x < -d \),

\[
V(x) = V(-d) = -\frac{\rho}{2\varepsilon} d^2
\]

Maximum field occurs at \( x = 0 \).

\[
|E_{max}| = \frac{\rho d}{\varepsilon} \leq E_b
\]

\[
\therefore \frac{\rho}{\varepsilon} \leq \frac{\varepsilon E_b}{d}
\]
Method 2: the mathematical way.

Poisson equation \( \nabla^2 V = -\frac{\rho(x)}{\varepsilon} \)

In this particular situation, \( \nabla^2 V = \frac{d^2 V}{dx^2} \)

Gauss's law: \( \nabla \cdot E = \frac{\rho}{\varepsilon} \)

In this particular situation: \( \nabla \cdot E = \frac{dE}{dx} \)

For \( 0 < d < x \).

\[
\frac{dE}{dx} = \frac{\rho}{\varepsilon} \quad \Rightarrow \quad E(x) = \frac{\rho}{\varepsilon} x + \text{Const.}
\]

Need a boundary condition to determine the Const.

No charge outside the slabs \( \Rightarrow \dot{E}(x > d) = 0 \)

\[
E(x) = \frac{\rho}{\varepsilon} (x-d) = -\frac{\rho}{\varepsilon} (d-x)
\]

For \( -d < x < 0 \).

\[
\frac{dE}{dx} = -\frac{\rho}{\varepsilon} \quad \Rightarrow \quad E(x) = -\frac{\rho}{\varepsilon} x + \text{Const}
\]

Similarly, using the boundary condition \( E(x \leq -d) = 0 \)

we get

\[
E(x) = -\frac{\rho}{\varepsilon} (d+x)
\]

Then take integrals to get \( V(x) \).