1. The top surface of an infinitely large ideal conductor plate is at the plane \( z = 0 \) of the Cartesian coordinate system. The conductor plate is grounded (i.e. at a potential 0). A positive point charge \( Q \) is located at \((0, 0, d/2)\). Assuming free space (i.e. vacuum, with permittivity \( \varepsilon_0 \)), find the following:

a) the electric field and potential at any point with \( z < 0 \). \( \text{ (5) } \)
b) the electric field and potential at any point on the \( z \) axis with \( z > 0 \) (You need to find the expression for the field \( E \) and potential \( V \) in terms of position \( z \)), and \( \text{ (5) } \)
c) the electric field and potential at point \((0, \sqrt{3}d, d/2)\). \( \text{ (10) } \)

Caution: The filed is a vector while the potential is a scalar. You may express a field in the form of \( E_x \hat{x} + E_y \hat{y} + E_z \hat{z} \).

\[ E = 0 \quad V = 0 \quad \text{for } z < 0. \]

b) Use the image method.

The image is \(-Q\) at \((0, 0, -\frac{d}{2})\)

For any point on the \( z \) axis, the field \( E(0, 0, z) = E_z \hat{z} \),

\[ E_z = \frac{Q}{4\pi \varepsilon_0 (3-d/2)^2} - \frac{Q}{4\pi \varepsilon_0 (3+d/2)^2} = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{1}{(3-d/2)^2} - \frac{1}{(3+d/2)^2} \right], \]

and the potential

\[ V(0, 0, z) = \frac{1}{4\pi \varepsilon_0} \left( \frac{1}{3-d/2} - \frac{1}{3+d/2} \right) \]

c) The field due to \(+Q\) is \( E_1 = E_x \hat{x} \), where

\[ E_x = \frac{Q}{4\pi \varepsilon_0 (\sqrt{3}d)^2} = \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{3d^2} \]

The field due to \(-Q\) is \( E_2 = E_{2y} \hat{y} + E_{2z} \hat{z} \)

\[ E_{2y} = \frac{Q}{4\pi \varepsilon_0 (2d)^2} = \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{4d^2} \]

\[ E_{2z} = -E_2 \cos 30^\circ = \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{4d^2} \cdot \frac{\sqrt{3}}{2}, \quad E_z = E_2 \sin 30^\circ = \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{8d^2} \]

\[ \hat{E} = E_1 + E_2 = \frac{Q}{4\pi \varepsilon_0} \left( \frac{1}{3d^2} - \frac{\sqrt{3}}{8d^2} \right) \hat{y} + \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{8d^2} \hat{z} \]

\[ = \frac{Q}{4\pi \varepsilon_0 d^2} \left[ \frac{1}{3} \hat{y} + \frac{\sqrt{3}}{8} \hat{z} \right] \]
\[ V = \frac{1}{4 \pi \varepsilon_0 (\sqrt{3}d)} - \frac{1}{4 \pi \varepsilon_0 (2d)} \]

\[ = \frac{1}{4 \pi \varepsilon_0 d} \left( \frac{1}{\sqrt{3}} - \frac{1}{2} \right) \]
2. A parallel wire transmission line is made of two wires of radius $a$ separated by a distance $d$, with the space filled by a dielectric with a relative permittivity $\varepsilon_r$ and a relative permeability of 1. To find the unit length capacitance $C'$ and the unit length inductance $L'$, consider two infinitely long parallel wires.

a) First, find the electric field due to one wire, at a distance $r$ from the axis of the wire, assuming that the charge on the wire is uniformly distributed on its surface. The linear charge density (i.e. charge per length) is $\rho_l$. You may assume $\rho_l$ is positive. \((5)\)

b) Assume the other wire, with a charge density $-\rho_l$, is at a distance $d$ from the first wire. Find the total electric field due to the two wires, at any point at a distance $x$ from the first wire along the line connecting the centers of the two wires, as shown in the figure. \((7.5)\)

c) Find the voltage $V$ between the two wires, and then use the relation $Q = CV$, where $Q$ is charge and $C$ is capacitance, to show that $C' = \frac{\pi \varepsilon_r \varepsilon_0}{\ln \frac{d}{a}}$, when $d \gg a$. \((7.5)\)

**Hint:** When $d \gg a$, the charge distribution on the wire surface does not matter, hence uniform distribution is a reasonable approximation. And, $d - a \approx d$.

**Hint:** Since the electrostatic field is conservative, you may choose any path to take the integral to find the voltage between two points.

\[
\begin{align*}
a) & \quad 2\pi r \int_0^l \rho_l \, dx = \rho_l \cdot l \quad \text{where} \quad l \text{ is the length of an arbitrary segment of the wire.} \\
& \quad \Rightarrow E = \frac{\rho_l}{2\pi \varepsilon_r \varepsilon_0} = \frac{\rho_l}{2\pi \varepsilon_r \varepsilon_0} r \\
& \quad \text{in the radial direction.} \\
b) & \quad E = E_1 + E_2 = \frac{\rho_l}{2\pi \varepsilon_r \varepsilon_0} \left( \frac{1}{x} + \frac{1}{d-x} \right) \\
c) & \quad V = \frac{\rho_l}{2\pi \varepsilon_r \varepsilon_0} \int_a^{d-a} \left( \frac{1}{x} + \frac{1}{d-x} \right) \, dx = \frac{\rho_l}{2\pi \varepsilon_r \varepsilon_0} \cdot 2 \int_a^{d-a} \frac{1}{x} \, dx \\
& \quad = \frac{\rho_l}{\pi \varepsilon_r \varepsilon_0} \ln \frac{d-a}{a} \approx \frac{\rho_l}{\pi \varepsilon_r \varepsilon_0} \ln \frac{d}{a} \\
\end{align*}
\]

\[
\begin{align*}
C' &= \frac{C}{l} = \frac{Q}{lV} = \frac{\rho_l}{\varepsilon_r \varepsilon_0} = \frac{\rho_l}{\pi \varepsilon_r \varepsilon_0} \ln \frac{d}{a}. \quad \text{Q.E.D.}
\end{align*}
\]
Problem 2 continued
(d) Find the Magnetic field due to one wire, at a distance \( r \) from the axis of the wire, assuming that the current \( I \) on the wire is uniformly distributed on its surface. (5)

e) Assume the other wire, carrying a current of equal magnitude \( I \) but in the opposite direction, is at a distance \( d \) from the first wire. Find the total magnetic field due to the two wires, at any point at a distance \( x \) from the first wire along the line connecting the centers of the two wires, as shown in the figure on the previous page. (5)

f) Find the total magnetic flux per unit length threading through the area between the two lines, and then use the relation \( \Phi = LI \), where \( \Phi \) is flux and \( L \) is inductance, to show that
\[
L' = \frac{\mu_0 I}{\pi} \ln \frac{d}{a}, \text{ when } d >> a
\]

\( d) \) Ampere's law \( \Rightarrow 2\pi r \left( \frac{B}{\mu_0} \right) = I \Rightarrow B = \frac{\mu_0 I}{2\pi r} \)

\( e) \ B = B_1 + B_2 = \frac{\mu_0 I}{2\pi} \left( \frac{1}{x} + \frac{1}{d-x} \right) \)

\( f) \ \Phi = \ell \int_{a}^{d-a} B \, dx \)
\[
= \ell \frac{\mu_0 I}{2\pi} \int_{a}^{d-a} \left( \frac{1}{x} + \frac{1}{d-x} \right) \, dx
\]
\[
= \ell \left( \frac{\mu_0 I}{\pi} \ln \frac{d-a}{a} \right) \approx \ell \left( \frac{\mu_0 I}{\pi} \ln \frac{d}{a} \right)
\]

\[
L' = \frac{L}{L} = \frac{\Phi}{\ell I} = \frac{\mu_0 I}{\pi} \ln \frac{d}{a} \quad \text{Q. E. D.}
\]
Problem 2 continued

g) Use the expressions given in c) and f) to find the characteristic impedance $Z_0$ for a parallel wire transmission line with $\varepsilon_r = 2.56$, $a = 0.89$ mm, and $d = 6.58$ mm. Round the result to an integer.  

$$\text{(Tips: } \sqrt{\mu_0/\varepsilon_0} = 120 \pi \Omega; \ 1.6^2 = 2.56)$$

h) If the transmission line is terminated in a load $Z_L = 60 \ \Omega$, find the voltage reflection coefficient $\Gamma_L$ at the load and the corresponding standing wave ratio $S$. 

i) Find the phase velocity $v_p$.  

$$\text{(Tips: } \sqrt{\varepsilon_r/\mu_0} = C = 3 \times 10^8 \text{ m/s})$$

j) Such a transmission line is 1.875 m long, fed by a step voltage at time $t = 0$ by a generator circuit with $V_g = 1$ V and $R_g = 450 \ \Omega$, and terminated in a load $Z_L = 60 \ \Omega$. Generate a bounce diagram for the voltage $v(x,t)$. 

k) Use the bounce diagram to plot $v(t)$ at the generator.

$$Z_0 = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{\mu_0}{\pi} \cdot \frac{\ln d}{a}} = \frac{1}{\pi} \sqrt{\frac{\mu_0}{\varepsilon_0} \cdot \frac{1}{\frac{\ln d}{a}}}$$

$$= \frac{1}{\pi} \cdot 120 \pi \cdot \frac{1}{1.6} \cdot \ln \frac{6.58}{0.89} = \frac{120}{1.6} \times 2 = 150 \ \Omega$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{60 - 150}{60 + 150} = -\frac{3}{7}$$

$$S = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1 + \frac{3}{7}}{1 - \frac{3}{7}} = \frac{10/7}{4/7} = 2.5$$

$$v_p = \frac{1}{\sqrt{L'(C')}} = \frac{1}{\sqrt{\left(\frac{\mu_0}{\pi} \cdot \frac{\ln d}{a}\right) \cdot \frac{\pi \varepsilon_0 \varepsilon_r}{2 \ln d} \cdot \frac{d}{a}}}$$

$$= \frac{C}{1.6}$$

j) The time of a single trip is

$$T = \frac{L}{v_p} = \frac{1.6 \cdot 1.875 \text{ m}}{C/1.6} = \frac{1.6 \times 1.875 \text{ m}}{3 \times 10^8 \text{ m/s}} = 10^{-8} \text{ s} = 10 \text{ ns}.$$
\[ V_i^+ = \frac{Z_o}{Z_o + R_g} \quad V_g = \frac{150}{150 + 450} \cdot 1V = 0.25V \]
3. When a TV antenna is very far from a TV station, the wave carrying the TV signal can be regarded as a plane wave. A circular-loop TV antenna with a $0.02 \text{ m}^2$ area is in the presence of a $f = 300 \text{ MHz}$ signal with a magnetic field $B = B_0 \sin(2\pi ft)$, where $t$ is time. When the loop is oriented in the $x$-$z$ plane, the loop antenna develops the maximum response with a peak response of $30 \text{ mV}$.

a) Determine $B_0$ and the direction (along which axis?) of the magnetic field. (10)

b) The TV station is in the $-z$ direction. Determine the direction (along which axis?) of the electric field and find the expression for the electric field (as a function of $t$). (5)

c) The $y$ axis points towards south. The earth’s magnetic field at the location of the antenna is about $30 \mu\text{T}$. Compare this value with $B_0$, and you will see it is orders of magnitude larger than $B_0$. Will the earth’s magnetic field overwhelm the TV signal? Why?

\[ S = 0.02 \text{ m}^2 \]
\[ \text{emf} = \frac{dB}{dt} \cdot S \]
\[ \frac{dB}{dt} = 2\pi f B \cos(2\pi ft) = \frac{\text{emf}}{S} \]

\[ \text{emf}_{\text{peak}} \]
\[ B_0 = \frac{S}{2\pi f} = \frac{30 \times 10^{-3}}{2\pi \times 300 \times 10^6 \times 0.02} \text{ T} \]
\[ = \frac{10}{4\pi} \times 10^{-9} \text{ T} = 0.8 \text{ nT} \]

It's direction is along the $y$ axis.

b) $\vec{k}$, propagation direction

\[ \vec{E} \text{ is along the } x \text{ axis.} \]
\[ E = c B = c B_0 \sin(2\pi ft) = 3 \times 10^8 \times 0.8 \times 10^{-9} \sin(6\pi \times 10^8 t) \frac{V}{m} \]
\[ = 0.24 \sin(6\pi \times 10^8 t) \text{ V/m} \]

c) No. Since the earth’s magnetic field does not change with time, it cannot induce an emf.