Gauss’s Law

The beauty of $1/R^2$

For a point charge,

$$\oint \mathbf{E} \cdot d\mathbf{s} = \frac{q}{4\pi\varepsilon_0} \frac{1}{R^2} \hat{\mathbf{R}} \cdot d\mathbf{s}$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{R^2} 4\pi R^2 = \frac{q}{\varepsilon_0}$$

a constant for any $R$.

For a chunk of charge,

$$\iiint \mathbf{E} \cdot d\mathbf{s} = \iiint \frac{\rho}{\varepsilon_0} \, dV = \frac{Q}{\varepsilon_0}$$

Integral form of Gauss's law: the "big picture"

“From our derivation you see that Gauss' law follows from the fact that the exponent in Coulomb's law is exactly two. A $1/r^3$ field, or any $1/r^n$ field with $n \neq 2$, would not give Gauss' law. So Gauss' law is just an expression, in a different form, of the Coulomb law...”

-- Richard Feynman
Electrostatics

Gauss's Law

Recall that all the \( \vec{E} \) lines must come out of positive charge, and end at negative charge. In other words, positive charge is the source of the \( \vec{E} \) field, and negative charge is the sink.

Now, let's see how to say this with the differential form.

Flux out of the cube thru face 1

\[
F_1 = -E_y \Delta x \Delta y \Delta z
\]

Flux out of the cube thru face 2

\[
F_2 = (E_y + \frac{\partial E_y}{\partial y}) \Delta x \Delta z
\]

The net flux:

\[
F_1 + F_2 = \frac{\partial E_y}{\partial y} \Delta y \Delta x \Delta z
\]

So, the total flux out is

\[
\left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta x \Delta y \Delta z
\]

\( \Delta x \Delta y \Delta z = \Delta V \)

Define as divergence of \( \vec{E} \):

\[
\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}
\]

\[
\nabla \cdot \vec{E} = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (\vec{E}_x \hat{x} + \vec{E}_y \hat{y} + \vec{E}_z \hat{z})
\]

\[
= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}
\]
For a small volume $\Delta V$,

$$\oint_{\Delta S} \vec{E} \cdot d\vec{S} = \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta V$$

$$= (\nabla \cdot \vec{E}) \Delta V$$

$$\therefore \nabla \cdot \vec{E} = \lim_{\Delta V \to 0} \frac{\oint_{\Delta S} \vec{E} \cdot d\vec{S}}{\Delta V}$$

$$\therefore \oint_{\Delta S} \vec{E} \cdot d\vec{S} = \oint (\nabla \cdot \vec{E}) \Delta V \propto \frac{1}{\epsilon_0} \int \rho \, dV$$

$$\varepsilon_0 \oint_{\Delta S} \vec{E} \cdot d\vec{S} = \oint \rho \, dV = Q$$

Gauss's law:

The charge is the source of the $\vec{E}$ field

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{D} = \varepsilon_0 \vec{E} \quad \Rightarrow \quad \nabla \cdot \vec{D} = \rho$$

$$\varepsilon_0 \oint_{\Delta S} \vec{E} \cdot d\vec{S} = \oint_{\Delta S} \vec{D} \cdot d\vec{S} = \oint (\nabla \cdot \vec{D}) \, dV = \oint \rho \, dV = Q$$

For a dielectric,...
Example:

Find the electric field due to an infinitely large sheet of charge \( \rho_s \) per unit area. It's a 2-D sheet, with a thickness.

By symmetry, the \( \vec{E} \) fields on the two sides of the sheet must be equal and opposite.

Also, \( \vec{E} \) is in the plane of the sheet.

Imagine a cylinder with area \( A \) and zero thickness.

If the cylinder is at the sheet,

\[
2 \varepsilon_0 E A = \rho_s A \quad \Rightarrow \quad E = \frac{\rho_s}{2 \varepsilon_0}
\]

Recall our result for the charged disk:

\[
E(z \to 0) = \frac{1}{2 \varepsilon_0} \cdot \frac{\Phi}{\pi a^2} = \frac{\rho_s}{2 \varepsilon_0}
\]

\( z \to 0 \) actually means \( \frac{z}{a} \to 0 \)

If the cylinder is elsewhere, the net flux is 0: what goes in comes out; no charge in the cylinder.
What if there are two charged sheets, one charged w/ $+\rho_s$ and the other $-\rho_s$?

$$E = \frac{\rho_s}{2\varepsilon_0} - \frac{\rho_s}{2\varepsilon_0} = 0$$

$$E = -\frac{\rho_s}{2\varepsilon_0} + \frac{\rho_s}{2\varepsilon_0} = -\frac{\rho_s}{\varepsilon_0}$$

$$E = -\frac{\rho_s}{2\varepsilon_0} + \frac{\rho_s}{2\varepsilon_0} = 0$$

This is the field distribution of a capacitor.

Another example for you to work out:

Two slabs of charge, w/ density $\rho$,
thickness $d$,
and infinite area

Find the field distribution.

Now, let's talk about dielectrics.
Electric Fields in Insulators (Dielectrics)

Polarization

1. Electronic polarization

\[ \vec{p} = q \vec{d} \]

2. Ionic polarization

3. Orientational polarization
**Polarization of the atoms of a dielectric material by a positive charge \( q \).**

\[
\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}
\]

\[
\rho_p = -\nabla \cdot \mathbf{P}
\]

\[
\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho_{\text{total}} = \frac{1}{\varepsilon_0} (\rho + \rho_p) = \frac{\rho}{\varepsilon_0} - \frac{1}{\varepsilon_0} \nabla \cdot \mathbf{P}
\]

\[
\varepsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} = \rho
\]

\[
\nabla \cdot (\varepsilon_0 + \chi_e \varepsilon_0) \mathbf{E} \equiv \nabla \cdot \mathbf{D} = \rho
\]

where \( \varepsilon \equiv \varepsilon_0 (1 + \chi_e) \equiv \varepsilon_0 \varepsilon_r \), \( \mathbf{D} \equiv \varepsilon_0 \varepsilon_r \mathbf{E} \equiv \varepsilon \mathbf{E} \)

\[
\mathbf{P} = \lim_{\Delta V \to 0} \frac{\sum \mathbf{p}_i}{\Delta V}
\]
Example

A charged dielectric sphere, charge density \( \rho \), dielectric constant \( \varepsilon_r \) \((\varepsilon = \varepsilon_r \varepsilon_0)\), radius \( R \).

Find the field distributions \( \mathbf{E}(r) \) & \( \mathbf{D}(r) \) for all \( r \) \((r \leq R \& R > r)\).

Solution:

For \( r \leq R \),

\[
4\pi r^2 \varepsilon E = \frac{4}{3} \pi r^3 \rho
\]

\[
\therefore \ E = \frac{1}{3\varepsilon} r \rho
\]

\[
D = \frac{1}{3} r \rho
\]

\[
E(R) = \frac{R \rho}{3\varepsilon}
\]

\[
D(R) = \frac{R \rho}{3}
\]
For $r > R$,

$$4\pi r^2 \varepsilon_0 E = \frac{4\pi R^3 \rho}{3} = \frac{Q}{4\pi r^2 \varepsilon_0}$$

$$E = \frac{Q}{4\pi r^2 \varepsilon_0} = \frac{R^3 \rho}{3\varepsilon_0 r^2}$$

Coulomb's law — as long as you have symmetry, ...

$$E(R) = \frac{R^3 \rho}{3\varepsilon_0 R^2} = \frac{R \rho}{3\varepsilon_0 R}$$

Unless $E = E_0$, there's a discontinuity.

Is this the right result?

Yes, you will understand this better after we discuss boundary conditions.

But for the moment, you only need to realize that the discontinuity is only in the $E$ field. If you consider the $D$ field, there's no discontinuity.