Consider a pair of wires when f is low, \( \lambda \ll l \).

Quasi static.

But, when f is high, \( \lambda \sim \omega r > l \).

Wave behavior.

You can actually get to this wave behavior by using circuit theory, w/o considering the details of the EM field!

There's capacitance between any two pieces of conductors.

\[ C = C' A z \]

\( C' \): capacitance per length
A piece of wire is actually an inductor

$$B \propto i$$

When $i$ changes with $t$, so does $B$.

$$\frac{di}{dt} \Rightarrow \frac{d^2i}{dt^2}$$

$$\mathbf{v} \times \mathbf{E} \propto \frac{\partial B}{\partial t} \Rightarrow \mathbf{v} \times \frac{di}{dt}$$

$$v = L \frac{di}{dt}$$

A pair of wires coupled but similar.

$$L = L' \Delta z$$

$L'$ is the inductance per length.

To make things simple, we first consider a pair of ideal wires:

No resistance, no shunt (leakage)
(Take good notes, different approach than in the textbook)

Now, zoom in on one segment.

\[ i(z, t) \xrightarrow{\Delta z} i(z + \Delta z, t) + i(z + \Delta z, t) \]

\[ \Delta u = u(z + \Delta z, t) - u(z, t) \]

\[ \frac{\partial u}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta u}{\Delta z} = -L' \frac{\partial i}{\partial t} \]

\[ \Delta i = -C' \Delta z \frac{\partial u}{\partial t} \]

\[ \frac{\partial^2 i}{\partial z^2} = -C' \frac{\partial^2 u}{\partial t \partial z} \]

\[ \frac{\partial^2 i}{\partial z \partial t} = -C' \frac{\partial^2 u}{\partial t^2} \]

\[ \frac{\partial^2 i}{\partial t^2} = L' C' \frac{\partial^2 u}{\partial t^2} \]

(When we write \( i \), we don't care if it's \( i(z) \) or \( i(z + \Delta z) \).

\( \Delta z \to 0 \)...)
What equation do \( v \) \& \( i \) follow?

Recall the wave equation from Physics 231?

Let \( v_p = \frac{1}{\sqrt{\varepsilon c}} \)

\[
\frac{\partial^2 v}{\partial z^2} = \frac{1}{v_p^2} \frac{\partial^2 v}{\partial t^2}
\]

The wave equation!

Is this amazing?!
You get the wave equation from the circuit theory! Why this circuit approach works?

There's something more amazing.

We don't go to the EM field details now, but just give you the expressions for \( \ell' \times \varepsilon' \); will revisit later. (Table 2-1 on pp. 53)

For a pair of wires

\[
\ell' = \frac{\mu}{\pi} \ln \left[ \frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]
\]

\[
\varepsilon' = \frac{\pi \varepsilon}{\ln \left[ \frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]}
\]

\[
\therefore \quad v_p = \frac{1}{\sqrt{\ell' \varepsilon'}} = \frac{1}{\sqrt{\mu \varepsilon}} \quad \text{consistent w/ EMF theory!}
\]

\( v = f(z - v_p t) \) is the general solution to this lossless, dispersionless wave equation.

Run the extra mile: Prove this.
The shape \( v = f(z - v_p t) \) will propagate i.e. translate, as long as the medium is lossless and dispersionless.
You can check this off-line for other two-conductor transmission lines.

Can you guess the solution to these wave equations?

\[ v(z, t) = V_0^+ \cos(\omega t - \beta z) \]
\[ i(z, t) = I_0^+ \cos(\omega t - \beta z) \]

\[ \frac{\omega}{k} = V_0 \]

A wave could also go the other way

\[ v(z, t) = V_0^- \cos(\omega t + \beta z) \]
\[ i(z, t) = I_0^- \cos(\omega t + \beta z) \]

Of course, any linear combo is a solution

\[ v(z, t) = V_0^+ \cos(\omega t - \beta z) + V_0^- \cos(\omega t + \beta z) \]
Same for \( i(z, t) \).

Remember we said last time that sometimes the \( \cos \) functions are a pain? We have a math tool — the phasor —
\[ v(t) = V_0^+ e^{j(wt - \beta z)} + V_0^- e^{j(wt + \beta z)} = \tilde{V}_0 e^{j\omega t} \]

\[ \tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \]

\[ \tilde{I}(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z} \]

This tool will make our life a lot easier when we deal more complicated situations.

Here's the more complicated situation:

There's no "ideal" wire. Any wire has some resistance.

\[ R = R' \cdot z \]

\[ R' : \text{resistance per length} \]

There's always some leakage between two conductors in a medium.

The shunt conductance

\[ G = G' \cdot z \]

\[ G' : \text{shunt conductance per length} \]

Notice that \( R' \) is the resistance of the conductors (or wires).

\[ G' \] is the shunt conductance between the conductors (wires).

\[ R' \neq V G' \]
Following the analysis in the text, which is similar to what we just did for the "ideal" case, you will get:

\[
\frac{d^2 \tilde{V}}{d\tilde{z}^2} - (R' + jwL') (G' + jwC') \tilde{V}(\tilde{z}) = 0
\]

Let's have a small digression back to the "ideal" case:

\[
R' = 0 \quad G' = 0
\]

\[
\frac{d^2 \tilde{V}}{d\tilde{z}^2} + \omega^2 L'C' \tilde{V} = 0
\]

\[
\tilde{V}(\tilde{z}) = V_0 e^{\theta + jw\sqrt{L'C'} \tilde{z}}
\]

Recall that

\[
\frac{\omega}{\beta} = \nu_p = \frac{1}{\sqrt{L'C'}} \quad \Rightarrow \quad \beta = \omega \sqrt{L'C'}
\]

\[
\therefore \tilde{V}(\tilde{z}) = V_0 e^{\pm \beta \tilde{z}} + j\beta \tilde{z}
\]

No surprise, this is the result we just got.
Now, back to the more complicated, real case. Let \( \nu^2 = (R' + jwL')(G' + jwC') \).

We then write

\[
\frac{d^2 \tilde{V}}{dz^2} - \nu^2 \tilde{V}(z) = 0
\]

\( \Rightarrow \tilde{V}(z) = V_0^+ e^{-\nu z} + V_0^- e^{\nu z} \)

Similarly, \( \tilde{I}(z) = I_0^+ e^{-\nu z} + I_0^- e^{\nu z} \).

\[\nu = \alpha + j\beta \quad \text{There are two } \nu \text{'s.} \]
\[\alpha = \text{Re}(\nu) \quad \beta = \text{Im}(\nu) \quad \text{Take the one } \]
\[\text{with positive } \alpha. \]

\[\tilde{V}(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{\alpha z} e^{j\beta z} \]

Talk about attenuation. \( R \& G \Rightarrow \text{loss} \).

Now, let's talk about a very important & interesting concept called characteristic impedance.
You may have heard about it. When you buy coax cables or flat lines, people say something like 50 \( \Omega \) or 75 \( \Omega \). What does that mean?

\[
\frac{d\tilde{V}}{dz} = -\nu V_0^+ e^{-\nu z} + \nu V_0^- e^{\nu z}
\]

\[-\frac{d\tilde{I}}{dz} = (R' + jwL') \tilde{I}(z) \]

\[= (R' + jwL') I_0^+ e^{-\nu z} + (R' + jwL') I_0^- e^{\nu z} \]

From circuit,
For the identity to hold for all $z$, 
\[ V_0^+ = (R' + jwL') I_0^+ \]
\[ \Rightarrow \frac{V_0^+}{I_0^+} = \frac{R' + jwL'}{Y} \]
\[ = \frac{R' + jwL'}{\sqrt{(R' + jwL')(G' + jwC')}} \]
\[ = \frac{R' + jwL'}{\sqrt{G' + jwC'}} \]

\[ y V_0^- = - (R' + jwL') I_0^- \]
\[ \Rightarrow \frac{V_0^-}{I_0^-} = - \frac{R' + jwL'}{Y} = - \sqrt{\frac{R' + jwL'}{G' + jwC'}} = - Z_0. \]

Define \( Z_0 = \sqrt{\frac{R' + jwL'}{G' + jwC'}} \) — characteristic impedance.

For the negative wave toward $+z$,
\[ \tilde{V}^+ = V_0^+ e^{-y_3} \]
\[ \tilde{I}^+ = I_0^+ e^{-y_3} \]

At any $z$,
\[ \frac{\tilde{V}^+(z)}{\tilde{I}^+(z)} = \frac{V_0^+}{I_0^+} = Z_0. \]
There's no way to tell the difference.
Every propagating away
Every dissipated.

Analogical: laser beam going to infinity
laser beam hitting a totally black absorber.

Impedance match

\[ Z_0 = \frac{V}{I} \]

The same as infinitely long.

For the wave traveling toward -z,
\[ \vec{V} = V_0 e^{j\beta z} \]
\[ \vec{E} = \vec{E}_0 e^{j\beta z} \]
\[ \frac{\vec{V}}{\vec{E}} = \frac{V_0}{E_0} = -Z_0 \]

Why the - sign?

\[ Z_0 = \sqrt{\frac{R'}{G' + j\omega C'}} \]

\[ R' = 0 \quad G' = 0 \]

\[ Z_0 = \sqrt{\frac{l''}{c'}} \]
If $Z_L = Z_0$, we say its impedance matched — the load gets all the energy from the wave.

What if $Z_L \neq Z_0$? What's gonna happen? The load doesn't get all the energy! Where does the rest of it go?

Consider a laser beam hitting a wall that's not totally black.

\[
\bar{V}_L = \bar{V}(z=0) = V_0^+ + V_0^-
\]

\[
\bar{I}_L = \bar{I}(z=0) = I_0^+ + I_0^- = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}
\]

By definition,

\[
Z_L = \frac{\bar{V}_L}{\bar{I}_L} = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right)Z_0
\]

Solve this, and you'll get
\[ V_0^- = \left( \frac{Z_L - Z_0}{Z_L + Z_0} \right) V_0^+ \]

Define voltage reflection coefficient
\[ \Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \]

\[ \frac{V_0^-}{V_0^+} = -\frac{V_0^-}{V_0^+} = -\Gamma \]

Recall that for a lossless line, \( Z_0 = \sqrt{\frac{L}{C}} \) is real.

But \( Z_L \) is usually complex.

Not that we want it to be... So, in general \( \Gamma \) is complex.

**Standing Wave**

A string with fixed ends.

Standing wave as opposed to traveling wave.

Now we show you that the standing wave is actually the superposition of an incident wave and the reflected wave.