Laplace System Analysis
System Descriptions

• Transfer functions of continuous-time systems can be found from analysis of
  – Differential Equations
  – Block Diagrams
  – Circuit Diagrams
System Descriptions

A circuit can be described by a system of differential equations:

$$-v_g(t) + R_1 \left[ i_L(t) + C \frac{d}{dt} (v_C(t)) \right] + L \frac{d}{dt} (i_L(t)) = 0$$

$$-L \frac{d}{dt} (i_L(t)) + v_C(t) + R_2 C \frac{d}{dt} (v_C(t)) = 0$$
System Descriptions

Using the Laplace transform, a circuit can be described by a system of algebraic equations

\[-V_g(s) + R_1[I_L(s) + sC V_C(s)] + sL I_L(s) = 0\]
\[-sL I_L(s) + V_C(s) + sR_2 C V_C(s) = 0\]
System Descriptions

A mechanical system can be described by a system of differential equations

\[ f(t) - K_d x'_1(t) - K_{s1} \left[ x_1(t) - x_2(t) \right] = m_1 x''_1(t) \]
\[ K_{s1} \left[ x_1(t) - x_2(t) \right] - K_{s2} x_2(t) = m_2 x''_2(t) \]

or a system of algebraic equations.

\[ F(s) - K_d s X_1(s) - K_{s1} \left[ X_1(s) - X_2(s) \right] = m_1 s^2 X_1(s) \]
\[ K_{s1} \left[ X_1(s) - X_2(s) \right] - K_{s2} X_2(s) = m_2 s^2 X_2(s) \]

\( f(t) \) is the system excitation signal and the velocity of mass, \( m_2 \), \( x'_2(t) \), is the system response signal, \( y(t) \).
System Descriptions

The mechanical system can also be described by a block diagram.

Time Domain

Frequency Domain
System Stability

System stability is very important. A continuous-time LTI system is stable if its impulse response is absolutely integrable. This translates into the frequency domain as the requirement that all the poles of the system transfer function must lie in the open left half of the $s$ plane (proven in the text). “Open left half-plane” means not including the $\omega$ axis.
System Interconnections

Cascade

\[ X(s) \xrightarrow{H_1(s)} X(s)H_1(s) \xrightarrow{H_2(s)} Y(s) = X(s)H_1(s)H_2(s) \]

\[ X(s) \xrightarrow{H_1(s)H_2(s)} Y(s) \]

Parallel

\[ X(s) \xrightarrow{H_1(s)} X(s)H_1(s) \xrightarrow{H_2(s)} X(s)H_2(s) \]

\[ X(s) \xrightarrow{H_1(s)+H_2(s)} Y(s) \]

\[ Y(s) = X(s)H_1(s) + X(s)H_2(s) = X(s)[H_1(s)+H_2(s)] \]
System Interconnections

**Feedback**

\[ E(s) \] Error signal

\[ H_1(s) \] Forward path transfer function of the "plant"

\[ H_2(s) \] Feedback path transfer function or the "sensor"

\[ T(s) = H_1(s)H_2(s) \]

Loop transfer function

\[ X(s) \]

\[ E(s) \]

\[ H_1(s) \]

\[ Y(s) \]

\[ H_2(s) \]

\[ E(s) = X(s) - H_2(s)Y(s) \]

\[ Y(s) = H_1(s)E(s) \]

\[ H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \]

\[ T(s) = H_1(s)H_2(s) \]

\[ H(s) = \frac{H_1(s)}{1 + T(s)} \]
Analysis of Feedback Systems

Beneficial Effects

\[ H(s) = \frac{K}{1 + KH_2(s)} \]

If \( K \) is large enough that \( KH_2(s) \gg 1 \) then \( H(s) \approx \frac{1}{H_2(s)} \).

This means that the overall system is the approximate inverse of the system in the feedback path. This kind of system can be useful for reversing the effects of another system.
Analysis of Feedback Systems

A very important example of feedback systems is an electronic amplifier based on an operational amplifier.

Let the operational amplifier gain be

\[ H_1(s) = \frac{V_o(s)}{V_e(s)} = -\frac{A_0}{1 - s/p} \]
Analysis of Feedback Systems

The amplifier can be modeled as a feedback system with this block diagram.

\[
\begin{align*}
V_i(s) & \quad \frac{Z_f(s)}{Z_i(s) + Z_f(s)} \quad + \quad V_e(s) \quad - \quad V_o(s) \\
& \quad - \frac{A_0}{1 - \frac{s}{p}} \\
& \quad - \frac{Z_i(s)}{Z_i(s) + Z_f(s)}
\end{align*}
\]

The overall gain can be written as

\[
\frac{V_o(s)}{V_i(s)} = -\frac{A_0 Z_f(s)}{(1 - s/p + A_0)Z_i(s) + (1 - s/p)Z_f(s)}
\]
Analysis of Feedback Systems

If the operational amplifier low-frequency gain $A_0$ is very large (which it usually is) then the overall amplifier gain reduces at low-frequencies to

$$\frac{V_0(s)}{V_i(s)} \approx -\frac{Z_f(s)}{Z_i(s)}$$

the gain formula based on an ideal operational amplifier.
Analysis of Feedback Systems

Let \( Z_f(s) = 10 \, \text{k}\Omega \) and let \( Z_i(s) = 1 \, \text{k}\Omega \) for a nominal gain of -10.
If \( A_0 = 10^7 \) and \( p = -100 \) then \( H(-j100) = -9.999989 + j0.000011 \)
If \( A_0 = 10^6 \) and \( p = -100 \) then \( H(-j100) = -9.99989 + j0.00011 \)
The change in overall system gain is about 0.001\% for a change in open-loop gain of a factor of 10.

The half-power bandwidth of the operational amplifier itself is 15.9 Hz (100/2\( \pi \)). The half-power bandwidth of the overall amplifier is approximately 14.5 MHz, an increase in bandwidth of a factor of approximately 910,000.
Analysis of Feedback Systems

Feedback can stabilize an unstable system. Let a forward-path transfer function be

\[ H_1(s) = \frac{1}{s - p}, \quad p > 0 \]

This system is unstable because it has a pole in the right half-plane. If we then connect feedback with a transfer function \( K \), a constant, the overall system gain becomes

\[ H(s) = \frac{1}{s - p + K} \]

and, if \( K > p \), the overall system is now stable.
Analysis of Feedback Systems

Feedback can make an unstable system stable but it can also make a stable system unstable. Even though all the poles of the forward and feedback systems may be in the open left half-plane, the poles of the overall feedback system can be in the right half-plane.

A familiar example of this kind of instability caused by feedback is a public address system. If the amplifier gain is set too high the system will go unstable and oscillate, usually with a very annoying high-pitched tone.
As the amplifier gain is increased, any sound entering the microphone makes a stronger sound from the speaker until, at some gain level, the returned sound from the speaker is as large as the originating sound into the microphone. At that point the system goes unstable.
Analysis of Feedback Systems

Stable Oscillation Using Feedback

Prototype Feedback System

Feedback System Without Excitation
Analysis of Feedback Systems

Stable Oscillation Using Feedback

Can the response be non-zero when the excitation is zero? Yes, if the overall system gain is infinite. If the system transfer function has a pole pair on the $\omega$ axis, then the transfer function is infinite at the frequency of that pole pair and there can be a response without an excitation. In practical terms the trick is to be sure the poles stay on the $\omega$ axis. If the poles move into the left half-plane the response attenuates with time. If the poles move into the right half-plane the response grows with time (until the system goes non-linear).
Analysis of Feedback Systems

Stable Oscillation Using Feedback

A real example of a system that oscillates stably is a laser. In a laser the forward path is an optical amplifier.

![Diagram of a laser with mirrors at each end]

The feedback action is provided by putting mirrors at each end of the optical amplifier.
Laser action begins when a photon is spontaneously emitted from the pumped medium in a direction normal to the mirrors.
If the “round-trip” gain of the combination of pumped laser medium and mirrors is unity, sustained oscillation of light will occur. For that to occur the wavelength of the light must fit into the distance between mirrors an integer number of times.
A laser can be modeled by a block diagram in which the $K$'s represent the gain of the pumped medium or the reflection or transmission coefficient at a mirror, $L$ is the distance between mirrors and $c$ is the speed of light.
Analysis of Feedback Systems

Root Locus

Common Type of Feedback System

System Transfer Function

\[ H(s) = \frac{K H_1(s)}{1 + K H_1(s) H_2(s)} \]

Loop Transfer Function

\[ T(s) = K H_1(s) H_2(s) \]
Analysis of Feedback Systems

Root Locus

Poles of $H(s)$ $\rightarrow$ Zeros of $1 + T(s)$

$T$ is of the form $\rightarrow$ $T(s) = K \frac{P(s)}{Q(s)}$

Poles of $H(s)$ $\rightarrow$ Zeros of $1 + K \frac{P(s)}{Q(s)}$

$\frac{Q(s) + KP(s)}{Q(s)} = 0$

$\frac{Q(s)}{K} + P(s) = 0$
Analysis of Feedback Systems

Root Locus

$K$ can range from zero to infinity. For $K$ approaching zero, using $Q(s) + KP(s) = 0$, the poles of $H$ are the same as the zeros of $Q(s) = 0$ which are the poles of $T$. For $K$ approaching infinity, using $Q(s)/K + P(s) = 0$ the poles of $H$ are the same as the zeros of $P(s) = 0$ which are the zeros of $T$. So the poles of $H$ start on the poles of $T$ and terminate on the zeros of $T$, some of which may be at infinity. The curves traced by these pole locations as $K$ is varied are called the root locus.
Analysis of Feedback Systems

Root Locus

Let \( H_1(s) = \frac{K}{(s+1)(s+2)} \) and let \( H_2(s) = 1 \).

Then \( T(s) = \frac{K}{(s+1)(s+2)} \)

No matter how large \( K \) gets this system is stable because the poles always lie in the left half-plane (although for large \( K \) the system may be very underdamped).
Analysis of Feedback Systems

Root Locus

Let \( H_1(s) = \frac{K}{(s+1)(s+2)(s+3)} \)

and let \( H_2(s) = 1 \)

At some finite value of \( K \) the system becomes unstable because two poles move into the right half-plane.
Analysis of Feedback Systems

Root Locus

The behavior of the zeros of polynomial equations as their coefficients vary has been studied by mathematicians and they have formulated rules obeyed by the movement of these zeros. In our case, the zeros of the denominator polynomial of the transfer function are the system poles and the movement traces out the root locus.
Analysis of Feedback Systems

Root Locus

Rules for Drawing a Root Locus

1. The number of branches in a root locus is the greater of the degree of the numerator and the denominator of \( T(s) \).
2. Each root-locus branch begins on a pole of \( T(s) \) and terminates on a zero of \( T(s) \). (Some zeros may be at infinity.)
3. Any portion of the real axis for which the sum of the number of real poles and/or real zeros lying to its right is odd, is a part of the root locus.
4. The root locus is symmetrical about the real axis.
Analysis of Feedback Systems

Root Locus

5. If the number of finite poles of $T(s)$ exceeds the number of finite zeros of $T(s)$ by an integer $m$ then $m$ branches of the root locus terminate on zeros of $T(s)$ which lie at infinity. Each of these branches approaches a straight-line asymptote and the angles of these asymptotes are at the angles, $(2k+1)\pi / m$, $k = 0, 1, \cdots m - 1$ with respect to the positive real axis. These asymptotes intersect on the real axis at the location,

$$\sigma = \frac{1}{m} \left( \sum \text{finite poles} - \sum \text{finite zeros} \right)$$

called the centroid of the root locus. (These are sums of all finite poles and all finite zeros, not just the ones on the real axis.)
Analysis of Feedback Systems

Root Locus

6. The breakaway or break-in points where the root locus branches intersect occur where \( \frac{d}{ds} \left( \frac{1}{T(s)} \right) = 0 \).
Analysis of Feedback Systems

Root Locus Examples

Let a feedback system have a loop transfer function

\[ T(s) = \frac{(s + 4)(s + 5)}{(s + 1)(s + 2)(s + 3)} \]

It has three finite poles at \( s = -1, -2 \) and \(-3\) and two finite zeros at \( s = -4 \) and \(-5\). There are three root locus branches (Rule 1). The allowed regions on the real axis are \(-2 < \sigma < -1\), \(-4 < \sigma < -3\) and \( \sigma < -5 \) (Rule 3).
The root locus must begin on the poles at -1, -2 and -3 and terminate on zeros at -4, -5 and infinity (Rule 2).
Analysis of Feedback Systems

Root Locus Examples

The two root locus branches beginning at -1 and -2 must move toward each other to stay in an allowed range (Rule 3). When they collide at a breakout point they both become complex and must be complex conjugates (Rule 4). The other branch beginning at -3 must move to the left to stay in an allowed range and can only terminate on the zero at -4 to maintain the symmetry of the root locus (Rules 3 and 4).
Analysis of Feedback Systems

Root Locus Examples
Analysis of Feedback Systems

Root Locus Examples

The other two root-locus branches must terminate on the zero at -5 and the zero at infinity. To maintain symmetry and approach the zeros in allowed regions, they must move to the allowed region on the real axis to the left of -5 (Rules 3 and 4). There is only one branch going to infinity and its angle is $\pi$ radians as it should be (Rule 5). The breakout and break-in points are found by solving

$$\frac{d}{ds}\left(\frac{1}{T(s)}\right) = \frac{d}{ds}\left[\frac{(s+1)(s+2)(s+3)}{(s+4)(s+5)}\right] = 0 \text{ (Rule 6).}$$

The solutions are $s = -9.47$, $-4.34$, $-2.69$ and $-1.5$. So the breakout point is at $-1.5$ and the break-in point is at $-9.47$. 
Analysis of Feedback Systems

Root Locus Examples

(The other solutions of \(\frac{d}{ds}\left(\frac{1}{T(s)}\right) = 0\) are the breakout and break-in points of the complementary root locus found by letting \(K\) approach negative infinity.)
Analysis of Feedback Systems

Root Locus Examples
A very common type of feedback system is the **unity-gain** feedback connection.

The aim of this type of system is to make the response “track” the excitation. When the error signal is zero, the excitation and response are equal.
Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

The Laplace transform of the error signal is

\[ E(s) = \frac{X(s)}{1 + H_1(s)}. \]

The steady-state value of this signal is (using the final-value theorem)

\[ \lim_{t \to \infty} e(t) = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{X(s)}{1 + H_1(s)} \]

If the excitation is the unit step \( u(t) \) then the steady-state error is

\[ \lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{1}{1 + H_1(s)} \]
Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

If the forward transfer function is in the common form,

\[ H_1(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_2 s^2 + b_1 s + b_0}{a_D s^D + a_{D-1} s^{D-1} + \cdots + a_2 s^2 + a_1 s + a_0} \]

then

\[ \lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{1}{1 + \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_2 s^2 + b_1 s + b_0}{a_D s^D + a_{D-1} s^{D-1} + \cdots + a_2 s^2 + a_1 s + a_0}} = \frac{a_0}{a_0 + b_0} \]

If \( a_0 = 0 \) and \( b_0 \neq 0 \) the steady-state error is zero and the forward transfer function can be written as

\[ H_1(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_2 s^2 + b_1 s + b_0}{s \left( a_D s^{D-1} + a_{D-1} s^{D-2} + \cdots + a_2 s + a_1 \right)} \]

which has a pole at \( s = 0 \).
Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

If the forward transfer function of a unity-gain feedback system has a pole at zero and the system is stable, the steady-state error with step excitation is zero. This type of system is called a “type 1” system (one pole at \( s = 0 \) in the forward transfer function). If there are no poles at \( s = 0 \), it is called a “type 0” system and the steady-state error with step excitation is non-zero.
Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

The steady-state error with ramp excitation is

- Infinite for a stable type 0 system
- Finite and non-zero for a stable type 1 system
- Zero for a stable type 2 system (2 poles at \( s = 0 \) in the forward transfer function)
System Responses to Standard Signals

Unit Step Response

Let \( H(s) = \frac{N(s)}{D(s)} \) be proper in \( s \). Then the Laplace transform of the unit-step response is

\[
Y(s) = H_{-1}(s) = \frac{N(s)}{sD(s)} = \frac{N_1(s)}{D(s)} + \frac{H(0)}{s}
\]

If the system is stable, the inverse Laplace transform of \( \frac{N(s)}{D(s)} \)

is called the transient response and the forced response is \( \frac{H(0)}{s} \).
System Responses to Standard Signals

Let \( H(s) = \frac{N(s)}{D(s)} \) be proper in \( s \). Then the Laplace transform of a general excitation is \( X(s) \) and the Laplace transform of the response is

\[
Y(s) = \frac{N(s)}{D(s)} X(s) = \frac{N(s)}{D(s)} \frac{N_x(s)}{D_x(s)} = \frac{N_1(s)}{D(s)} + \frac{N_{x1}(s)}{D_x(s)}
\]

same poles as system
same poles as excitation
System Responses to Standard Signals

Unit Step Response

Let \( H(s) = \frac{A}{1 - s/p} \).

Then the unit-step response is

\[ h_{-1}(t) = A\left(1 - e^{pt}\right)u(t). \]
System Responses to Standard Signals
Unit Step Response

Let

\[ H(s) = \frac{A\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \]
System Responses to Standard Signals

Unit Step Response

Let \( H(s) = \frac{A\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \)
System Responses to Standard Signals

Let \( H(s) = \frac{N(s)}{D(s)} \) be proper in \( s \). If the excitation is a unit-amplitude cosine applied to the system at time \( t = 0 \), the response is

\[
Y(s) = \frac{N(s)}{D(s)} \frac{s}{s^2 + \omega_0^2}
\]

which can be reduced and inverse Laplace transformed into

\[
y(t) = \mathcal{L}^{-1} \left( \frac{N_1(s)}{D(s)} \right) + |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0)) u(t)
\]

If the system is stable, the steady-state response is a sinusoid of same frequency as the excitation but, generally a different magnitude and phase.
Standard Realizations of Systems

There are multiple ways of drawing a system block diagram corresponding to a given transfer function of the form

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{N} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

(Here the numerator and denominator are both assumed to be of order N. If the numerator order is less than N, some of the b’s are zero.)
Standard Realizations of Systems
Cascade Form

The transfer function can be factored into the form,

\[
H(s) = A \frac{s - z_1}{s - p_1} \frac{s - z_2}{s - p_2} \ldots \frac{s - z_M}{s - p_M} \frac{1}{s - p_{M+1}} \frac{1}{s - p_{M+2}} \ldots \frac{1}{s - p_N}
\]

and each factor can be realized in a small Direct Form II subsystem of either of the two forms

and these subsystems can then be cascade connected.
Standard Realizations of Systems

Cascade Form

A problem that arises in the cascade form is that some poles or zeros may be complex. In that case, a complex conjugate pair can be combined into one **second-order subsystem** of the form.
Standard Realizations of Systems
Parallel Form

The transfer function can be expanded in partial fractions of the form

\[ H(s) = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \ldots + \frac{K_N}{s - p_N} \]

Each of these terms describes a subsystem. When all the subsystems are connected in parallel the overall system is realized.
Standard Realizations of Systems

Parallel Form

\[ X(s) \rightarrow \sum_{i=1}^{N} K_i S^{-1} - p_i \rightarrow Y(s) \]
8. In the system in Figure E.8, \( x_c(t) = \text{sinc}(t) \), \( f_c = 10 \) and the cutoff frequency of the lowpass filter is 1 Hz. Graph the signals \( x_c(t) \), \( y_c(t) \), \( y_d(t) \) and \( y_f(t) \) and the magnitudes and phases of their CTFTs.

9. A sinusoid \( x(t) = A_m \cos(2\pi f_c t) \) modulates a sinusoidal carrier \( A_c \cos(2\pi f_c t) \) in a double-sideband transmitted-carrier (DSB-TC) system of the type illustrated in Figure E.9. If \( A_m = 1 \), \( f_m = 10 \), \( A_c = 4 \), \( f_c = 1000 \) and \( m = 1 \), find the numerical value of the total signal power in \( y(t) \) at the carrier frequency \( P_c \), and the numerical value of the total signal power in \( y(t) \) in its sidebands \( P_s \).

12. A power signal \( x(t) \) with no signal power outside the frequency range \(-f_c/100 < f < f_c/100 \) is multiplied by a carrier \( \cos(2\pi f_c t) \) to form a signal, \( y_c(t) \). Then \( y_c(t) \) is multiplied by \( \cos(2\pi f_c t) \) to form \( y_f(t) \). Then \( y_f(t) \) is filtered by an ideal lowpass filter whose frequency response is \( H(f) = 6 \text{rect}(f/2f_c) \) to form \( y_f(t) \). What is the ratio of the signal power in \( y_f(t) \) to the signal power in \( x(t) \)? \( P_{y_f}/P_x \)?

14. In a PM modulator let the information signal be \( x(t) = \sin(10^5 t) \), let the carrier be \( \cos(2\pi \times 10^6 t) \) and let the modulation indices be \( k_f = \pi/5 \) and \( k_f = k_g \times 10^6/15 \). Graph the modulator output signal for the time range \( 0 < t < 20 \mu s \). Compute the modulator output two ways, (1) directly as a modulated signal, and (2) using the narrowband PM and FM approximations. Compare the graphs.

15. In Figure E.15 is a circuit diagram of an envelope detector. Model the diode as ideal and let the input voltage signal be a cosine at 100 kHz with an amplitude of 200 mV. Let the RC time constant be 60 microseconds. Find and graph the magnitude of the CTFT of the output voltage signal.

![Figure E.8](image_url)  
**Figure E.8**

![Figure E.9](image_url)  
**Figure E.9**

![Figure E.11](image_url)  
**Figure E.11**

![Figure E.15](image_url)  
**Figure E.15** An envelope detector.
Due date: Nov. 11, 2016

15. Find the $s$-domain transfer functions for the circuits in Figure E.15 and then draw block diagrams for them as systems with $v_x(s)$ as the excitation and $v_y(s)$ as the response.

![Circuit diagrams](image_url)

**Figure E.15**

16. Determine whether the systems with these transfer functions are stable, marginally stable or unstable.

(a) $H(s) = \frac{s^2 + 2}{s^2 + 8}$

(b) $H(s) = \frac{s^2 - 2}{s^2 + 8}$

(c) $H(s) = \frac{s^2}{s^2 + 4s + 8}$

(d) $H(s) = \frac{s^2}{s^2 - 4s + 8}$

(e) $H(s) = \frac{s}{s^2 + 4s^2 + 8}$

18. Find the expression for the overall system transfer function of the system in Figure E.18. For what positive values of $\kappa$ is the system stable?

![Block diagram](image_url)

**Figure E.18**

20. A classical example of the use of feedback is the phase-locked loop used to demodulate frequency-modulated signals (Figure E.20).

![Phase-locked loop diagram](image_url)

**Figure E.20**

The input signal $x(t)$ is a frequency-modulated sinusoid. The phase detector detects the phase difference between the input signal and the signal produced by the voltage-controlled oscillator. The response of the phase detector is a voltage proportional to phase difference. The loop filter filters that voltage. Then the loop filter output signal controls the frequency of the voltage-controlled oscillator. When the input signal is at a constant frequency and the loop is "locked" the phase difference between the two phase-detector input signals is zero. (In an actual phase detector the phase difference is 90° at lock. But that is not significant in this analysis since that only causes a 90° phase shift and has no impact on system performance or stability.) As the frequency of the input signal $x(t)$ varies, the loop detects the accompanying phase variation and tracks it. The overall output signal $y(t)$ is a signal proportional to the frequency of the input signal.

The actual excitation, in a system sense, of this system is not $x(t)$, but rather the phase of $x(t)$, $\phi_x(t)$, because the phase detector detects differences in phase, not voltage. Let the frequency of $x(t)$ be $f_0$. The relation between phase and frequency can be seen by examining a sinusoid. Let $x(t) = A \cos(2\pi f_0 t)$. The phase of this cosine is $2\pi f_0 t$ and, for a simple sinusoid ($f_0$ constant), it increases linearly with time. The frequency is $f_0$, the derivative of the phase. Therefore the relation between phase and frequency for a frequency-modulated signal is

$$f(t) = \frac{1}{2\pi} \frac{d}{dt} \phi_x(t).$$

Let the frequency $x(t)$ be 100 MHz. Let the transfer function of the voltage-controlled oscillator be $10^7$. Let the transfer function of the loop filter be $H(s) = \frac{1}{s + 1 \times 10^7}$. Let the transfer function of the phase detector be $\frac{1}{\sqrt{s}}$. If the frequency of $x(t)$ signal suddenly changes to 100.001 MHz, graph the change in the output signal, $\Delta y(t)$. 

Let the transfer function of the phase detector be $\frac{1}{\sqrt{s}}$. If the frequency of $x(t)$ signal suddenly changes to 100.001 MHz, graph the change in the output signal, $\Delta y(t)$.