The $z$ Transform
Generalizing the DTFT

The forward DTFT is defined by \( X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \) in which \( \Omega \) is discrete-time radian frequency, a real variable. The quantity \( e^{j\Omega n} \) is then a complex sinusoid whose magnitude is always one and whose phase can range over all angles. It always lies on the unit circle in the complex plane. If we now replace \( e^{j\Omega} \) with a variable \( z \) that can have any complex value we define the \( z \) transform \( X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \).

The DTFT expresses signals as linear combinations of complex sinusoids. The \( z \) transform expresses signals as linear combinations of complex exponentials.
Complex Exponential Excitation

Let the excitation of a discrete-time LTI system be a complex exponential of the form $Az^n$ where $z$ is, in general, complex and $A$ is any constant. Using convolution, the response $y[n]$ of an LTI system with impulse response $h[n]$ to a complex exponential excitation $x[n]$ is

$$y[n] = h[n] * Az^n = A \sum_{m=-\infty}^{\infty} h[m]z^{n-m} = Az^n \sum_{m=-\infty}^{\infty} h[m]z^{-m}$$

The response is the product of the excitation and the $z$ transform of $h[n]$ defined by $H(z) = \sum_{m=-\infty}^{\infty} h[n]z^{-n}$. 
The Transfer Function

If an LTI system with impulse response \( h[n] \) is excited by a signal, \( x[n] \), the \( z \) transform \( Y(z) \) of the response \( y[n] \) is

\[
Y(z) = \sum_{n=\infty}^{\infty} y[n]z^{-n} = \sum_{n=\infty}^{\infty} (h[n] \ast x[n])z^{-n} = \sum_{n=\infty}^{\infty} \sum_{m=\infty}^{\infty} h[m]x[n-m]z^{-n}
\]

\[
Y(z) = \sum_{m=\infty}^{\infty} h[m] \sum_{n=\infty}^{\infty} x[n-m]z^{-n}
\]

Let \( q = n-m \). Then

\[
Y(z) = \sum_{m=\infty}^{\infty} h[m] \sum_{q=\infty}^{\infty} x[q]z^{-(q+m)} = \sum_{m=\infty}^{\infty} h[m]z^{-m} \sum_{q=\infty}^{\infty} x[q]z^{-q}
\]

\[
= \underbrace{\sum_{m=\infty}^{\infty} h[m]z^{-m}}_{=H(z)} \underbrace{\sum_{q=\infty}^{\infty} x[q]z^{-q}}_{=X(z)}
\]

\[
Y(z) = H(z)X(z)
\]

\( H(z) \) is the transfer function.
Systems Described by Difference Equations

The most common description of a discrete-time system is a difference equation of the general form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

It was shown in Chapter 5 that the transfer function for a system of this type is

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

or

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_{M-1} z + b_M}{a_0 z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}$$
Direct Form II Realization

Direct Form II realization of a discrete-time system is similar in form to Direct Form II realization of continuous-time systems.

A continuous-time system can be realized with integrators, summing junctions and multipliers.

A discrete-time system can be realized with delays, summing junctions and multipliers.
Direct Form II Realization
The Inverse $z$ Transform

The inversion integral is

$$x[n] = \frac{1}{j2\pi} \oint_C X(z) z^{n-1} dz.$$ 

This is a contour integral in the complex plane and is beyond the scope of this course. The notation $x[n] \leftarrow\mathcal{Z} \rightarrow X(z)$ indicates that $x[n]$ and $X(z)$ form a "$z$-transform pair".
Existence of the $z$ Transform

Time Limited Signals

If a discrete-time signal $x[n]$ is time limited and bounded, the $z$ transformation summation $\sum_{n=-\infty}^{\infty} x[n] z^{-n}$ is finite and the $z$ transform of $x[n]$ exists for any non-zero value of $z$. 
Existence of the $z$ Transform

Right- and Left-Sided Signals

A right-sided signal $x_r[n]$ is one for which $x_r[n] = 0$ for any $n < n_0$ and a left-sided signal $x_l[n]$ is one for which $x_l[n] = 0$ for any $n > n_0$. 
Existence of the $z$ Transform

Right- and Left-Sided Exponentials

$x[n] = \alpha^n u[n-n_0]$, $\alpha \in \mathbb{C}$

$x[n] = \beta^n u[n_0-n]$, $\beta \in \mathbb{C}$
Existence of the $z$ Transform

The $z$ transform of $x[n] = \alpha^n u[n - n_0]$, $\alpha \in \mathbb{C}$ is

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n - n_0] z^{-n} = \sum_{n=n_0}^{\infty} (\alpha z^{-1})^n$$

if the series converges and it converges if $|z| > |\alpha|$. The path of integration of the inverse $z$ transform must lie in the region of the $z$ plane outside a circle of radius $|\alpha|$. 

12/29/10

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Existence of the $z$ Transform

The $z$ transform of $x[n] = \beta^n u[n_0 - n]$, $\beta \in \mathbb{C}$ is

$$X(z) = \sum_{n=-\infty}^{n_0} \beta^n z^{-n} = \sum_{n=-\infty}^{n_0} (\beta z^{-1})^n = \sum_{n=-n_0}^{\infty} (\beta^{-1}z)^n$$

if the series converges and it converges if $|z| < |\beta|$. The path of integration of the inverse $z$ transform must lie in the region of the $z$ plane inside a circle of radius $|\beta|$. 

![Path of Integration](image)
Existence of the $z$ Transform

$x[n] = (1.2)^n u[n] + (3)^n u[-n-1]$

ROC is $1.2 < |z| < 3$

$x[n] = (0.95)^n u[n] + (0.9)^n u[-n-1]$

No ROC

$x[n] = (0.85)^n \cos(2\pi n/6) u[n] + (0.9)^n \cos(2\pi n/6) u[-n-1]$

ROC is $0.85 < |z| < 0.9$

$x[n] = (1.1)^n \cos(2\pi n/6) u[n] + (1.05)^n \cos(2\pi n/6) u[-n-1]$

No ROC
Some Common $z$ Transform Pairs

$$\delta[n] \xrightarrow{z} 1, \text{ All } z$$

$$u[n] \xrightarrow{z} \frac{z}{z-1}, |z| > 1$$

$$\alpha^n u[n] \xrightarrow{z} \frac{z}{z-\alpha}, |z| > |\alpha|$$

$$n u[n] \xrightarrow{z} \frac{z}{(z-1)^2}, |z| > 1$$

$$n \alpha^n u[n] \xrightarrow{z} \frac{\alpha z}{(z-\alpha)^2}, |z| > |\alpha|$$

$$\sin(\Omega_0 n) u[n] \xrightarrow{z} \frac{z \sin(\Omega_0)}{z^2 - 2z \cos(\Omega_0) + 1}, |z| > 1$$

$$\cos(\Omega_0 n) u[n] \xrightarrow{z} \frac{z [z - \cos(\Omega_0)]}{z^2 - 2z \cos(\Omega_0) + 1}, |z| > 1$$

$$\alpha^n \sin(\Omega_0 n) u[n] \xrightarrow{z} \frac{z \alpha \sin(\Omega_0)}{z^2 - 2\alpha \cos(\Omega_0) + \alpha^2}, |z| > |\alpha|$$

$$\alpha^n \cos(\Omega_0 n) u[n] \xrightarrow{z} \frac{z [z - \alpha \cos(\Omega_0)]}{z^2 - 2\alpha \cos(\Omega_0) + \alpha^2}, |z| > |\alpha|$$

$$a^n \xrightarrow{z} \frac{z}{z-\alpha}, |\alpha| < |z| < |\alpha^{-1}|$$

$$u[n-n_0] - u[n-n_1] \xrightarrow{z} \frac{z}{z-1} (z^{-n_0} - z^{-n_1}) = \frac{z^{n_1-n_0} + z^{n_1-n_2} + \cdots + z + 1}{z^{n_1}}$$

$$, |z| > 0$$
z-Transform Properties

Given the z-transform pairs \( g[n] \xleftarrow{\mathcal{Z}} G(z) \) and \( h[n] \xleftarrow{\mathcal{Z}} H(z) \) with ROC's of \( \text{ROC}_G \) and \( \text{ROC}_H \) respectively the following properties apply to the z transform.

**Linearity**
\[
\alpha g[n] + \beta h[n] \xleftarrow{\mathcal{Z}} \alpha G(z) + \beta H(z)
\]
\( \text{ROC} = \text{ROC}_G \cap \text{ROC}_H \)

**Time Shifting**
\[
g[n - n_0] \xleftarrow{\mathcal{Z}} z^{-n_0} G(z)
\]
\( \text{ROC} = \text{ROC}_G \) except perhaps \( z = 0 \) or \( z \to \infty \)

**Change of Scale in z**
\[
\alpha^n g[n] \xleftarrow{\mathcal{Z}} G(\frac{z}{\alpha})
\]
\( \text{ROC} = |\alpha|\text{ROC}_G \)
$z$-Transform Properties

**Time Reversal**

\[ g[-n] \xrightarrow{\mathcal{Z}} G(z^{-1}) \]

\[ \text{ROC} = \frac{1}{\text{ROC}_G} \]

**Time Expansion**

\[ \begin{cases} g[n/k] & \text{, } n/k \text{ and integer} \\ 0 & \text{, otherwise} \end{cases} \xrightarrow{\mathcal{Z}} G(z^k) \]

\[ \text{ROC} = \left(\text{ROC}_G\right)^{1/k} \]

**Conjugation**

\[ g^*[n] \xrightarrow{\mathcal{Z}} G^*(z^*) \]

\[ \text{ROC} = \text{ROC}_G \]

**$z$-Domain Differentiation**

\[ -ng[n] \xrightarrow{\mathcal{Z}} z \frac{d}{dz} G(z) \]

\[ \text{ROC} = \text{ROC}_G \]
$z$-Transform Properties

**Convolution**
\[ g[n] \ast h[n] \xrightarrow{\mathcal{Z}} H(z)G(z) \]

**First Backward Difference**
\[ g[n] - g[n-1] \xrightarrow{\mathcal{Z}} (1 - z^{-1})G(z) \]
\[ \text{ROC} \supseteq \text{ROC}_G \cap |z| > 0 \]

**Accumulation**
\[ \sum_{m=-\infty}^{n} g[m] \xrightarrow{\mathcal{Z}} \frac{z}{z-1}G(z) \]
\[ \text{ROC} \supseteq \text{ROC}_G \cap |z| > 1 \]

**Initial Value Theorem**
If \( g[n] = 0 \), \( n < 0 \) then \( g[0] = \lim_{z \to \infty} G(z) \)

**Final Value Theorem**
If \( g[n] = 0 \), \( n < 0 \), \( \lim_{n \to \infty} g[n] = \lim_{z \to 1} (z-1)G(z) \)
if \( \lim_{n \to \infty} g[n] \) exists.
$z$-Transform Properties

For the final-value theorem to apply to a function $G(z)$ all the finite poles of the function $(z - 1)G(z)$ must lie in the open interior of the unit circle of the $z$ plane. Notice this does not say that all the poles of $G(z)$ must lie in the open interior of the unit circle. $G(z)$ could have a single pole at $z = 1$ and the final-value theorem could still apply.
The Inverse $z$ Transform

Synthetic Division

For rational $z$ transforms of the form
\[
H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}
\]
we can always find the inverse $z$ transform by synthetic division. For example,

\[
H(z) = \frac{(z - 1.2)(z + 0.7)(z + 0.4)}{(z - 0.2)(z - 0.8)(z + 0.5)} \quad , \quad |z| > 0.8
\]

\[
H(z) = \frac{z^3 - 0.1 z^2 - 1.04 z - 0.336}{z^3 - 0.5 z^2 - 0.34 z + 0.08} \quad , \quad |z| > 0.8
\]
The Inverse $z$ Transform

Synthetic Division

\[ z^3 - 0.5z^2 - 0.34z + 0.08 \overline{) z^3 - 0.1z^2 - 1.04z - 0.336} \]
\[ z^3 - 0.5z^2 - 0.34z + 0.08 \overline{) 0.4z^2 - 0.7z - 0.256} \]
\[ 0.4z^2 - 0.2z - 0.136 - 0.032z^{-1} \]
\[ 0.5z - 0.12 + 0.032z^{-1} \]
\[ \vdots \]

The inverse $z$ transform is

\[ \delta[n] + 0.4\delta[n-1] + 0.5\delta[n-2] \cdots \xrightarrow{\mathcal{Z}} 1 + 0.4z^{-1} + 0.5z^{-2} \cdots \]
The Inverse \( z \) Transform

Synthetic Division

We could have done the synthetic division this way.

\[
\begin{array}{r}
0.08 - 0.34z - 0.5z^2 + z^3 \\
\hline
-0.336 - 1.04z - 0.1z^2 + z^3 \\
\hline
-0.336 + 1.428z + 2.1z^2 - 4.2z^3 \\
\hline
-2.468z - 2.2z^2 + 5.2z^3 \\
\hline
-2.468z + 10.489z^2 + 15.425z^3 - 30.85z^4 \\
\hline
-12.689z^2 - 10.225z^3 + 30.85z^4 \\
\hline
\end{array}
\]

\( -4.2 - 30.85z - 158.613z^2 \cdots \)

\( -4.2 \delta[n] - 30.85 \delta[n+1] - 158.613 \delta[n+2] \cdots \xleftarrow{\mathcal{F}} -4.2 - 30.85z - 158.613z^2 \cdots \)

but with the restriction \( |z| > 0.8 \) this second form does not converge and is therefore not the inverse \( z \) transform.
The Inverse $z$ Transform

Synthetic Division

We can always find the inverse $z$ transform of a rational function with synthetic division but the result is not in closed form. In most practical cases a closed-form solution is preferred.
Partial Fraction Expansion

Partial-fraction expansion works for inverse $z$ transforms the same way it does for inverse Laplace transforms. But there is a situation that is quite common in inverse $z$ transforms which deserves mention. It is very common to have $z$-domain functions in which the number of finite zeros equals the number of finite poles (making the expression improper in $z$) with at least one zero at $z = 0$.

$$H(z) = \frac{z^{N-M}(z - z_1)(z - z_2) \cdots (z - z_M)}{(z - p_1)(z - p_2) \cdots (z - p_N)}$$
Partial Fraction Expansion

Dividing both sides by $z$ we get

$$H(z) = \frac{z^{N-M-1}(z-z_1)(z-z_2)\cdots(z-z_M)}{z(z-p_1)(z-p_2)\cdots(z-p_N)}$$

and the fraction on the right is now proper in $z$ and can be expanded in partial fractions.

$$H(z) = \frac{K_1}{z-p_1} + \frac{K_2}{z-p_2} + \cdots + \frac{K_N}{z-p_N}$$

Then both sides can be multiplied by $z$ and the inverse transform can be found.

$$H(z) = \frac{zK_1}{z-p_1} + \frac{zK_2}{z-p_2} + \cdots + \frac{zK_N}{z-p_N}$$

$$h[n] = K_1p_1^n u[n] + K_2p_2^n u[n] + \cdots + K_Np_N^n u[n]$$
\textbf{z-Transform Properties}

An LTI system has a transfer function

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1/2}}{z^2 - z + 2/9}, \quad \lvert z \rvert > 2/3 \]

Using the time-shifting property of the \( z \) transform draw a block diagram realization of the system.

\[ Y(z)(z^2 - z + 2/9) = X(z)(z - 1/2) \]

\[ z^2 Y(z) = z X(z) - (1/2) X(z) + z Y(z) - (2/9) Y(z) \]

\[ Y(z) = z^{-1} X(z) - (1/2) z^{-2} X(z) + z^{-1} Y(z) - (2/9) z^{-2} Y(z) \]
**z-Transform Properties**

\[
Y(z) = z^{-1} X(z) - \left(\frac{1}{2}\right) z^{-2} X(z) + z^{-1} Y(z) - \left(\frac{2}{9}\right) z^{-2} Y(z)
\]

Using the time-shifting property

\[
y[n] = x[n-1] - \left(\frac{1}{2}\right) x[n-2] + y[n-1] - \left(\frac{2}{9}\right) y[n-2]
\]
**z-Transform Properties**

Let \( g[n] \xrightarrow{\mathcal{Z}} G(z) = \frac{z-1}{(z-0.8e^{-j\pi/4})(z-0.8e^{j\pi/4})} \). Draw a pole-zero diagram for \( G(z) \) and for the \( z \) transform of \( e^{j\pi n/8} g[n] \).

The poles of \( G(z) \) are at \( z = 0.8 e^{\pm j\pi/4} \) and its single finite zero is at \( z = 1 \). Using the change of scale property

\[
e^{j\pi n/8} g[n] \xrightarrow{\mathcal{Z}} G\left(ze^{-j\pi/8}\right) = \frac{ze^{-j\pi/8} - 1}{\left(ze^{-j\pi/8} - 0.8e^{-j\pi/4}\right)\left(ze^{-j\pi/8} - 0.8e^{j\pi/4}\right)}
\]

\[
G\left(ze^{-j\pi/8}\right) = \frac{e^{-j\pi/8}(z-e^{j\pi/8})}{\left(e^{-j\pi/8}(z-0.8e^{-j\pi/8})\right)e^{-j\pi/8}\left(z-0.8e^{j3\pi/8}\right)}
\]

\[
G\left(ze^{-j\pi/8}\right) = e^{j\pi/8}\frac{z-e^{j\pi/8}}{\left(z-0.8e^{-j\pi/8}\right)\left(z-0.8e^{j3\pi/8}\right)}
\]
$z$-Transform Properties

$G(ze^{-j\pi/8})$ has poles at $z = 0.8e^{-j\pi/8}$ and $0.8e^{+j3\pi/8}$ and a zero at $z = e^{j\pi/8}$. All the finite zero and pole locations have been rotated in the $z$ plane by $\pi/8$ radians.
$z$-Transform Properties

Using the accumulation property and $u[n] \leftrightarrow \mathcal{Z} \rightarrow \frac{z}{z-1}, \quad |z| > 1$

show that the $z$ transform of $nu[n]$ is $\frac{z}{(z-1)^2}, \quad |z| > 1$.

$$nu[n] = \sum_{m=0}^{n} u[m-1]$$

$$u[n-1] \leftrightarrow \mathcal{Z} \rightarrow z^{-1} \frac{z}{z-1} = \frac{1}{z-1}, \quad |z| > 1$$

$$nu[n] = \sum_{m=0}^{n} u[m-1] \leftrightarrow \mathcal{Z} \rightarrow \left(\frac{z}{z-1}\right) \frac{1}{z-1} = \frac{z}{(z-1)^2}, \quad |z| > 1$$
Inverse $z$ Transform Example

Find the inverse $z$ transform of

$$X(z) = \frac{z}{z-0.5} - \frac{z}{z+2}, \quad 0.5 < |z| < 2$$

Right-sided signals have ROC’s that are outside a circle and left-sided signals have ROC’s that are inside a circle. Using

$$\alpha^n u[n] \xrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

$$-\alpha^n u[-n-1] \xrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}}, \quad |z| < |\alpha|$$

We get

$$(0.5)^n u[n] + (-2)^n u[-n-1] \xrightarrow{\mathcal{Z}} X(z) = \frac{z}{z-0.5} - \frac{z}{z+2}, \quad 0.5 < |z| < 2$$
Inverse $z$ Transform Example

Find the inverse $z$ transform of

$$X(z) = \frac{z}{z - 0.5} - \frac{z}{z + 2}, \quad |z| > 2$$

In this case, both signals are right sided. Then using

$$\alpha^n u[n] \xleftarrow{\mathcal{Z}} \frac{z}{z - \alpha} = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|$$

We get

$$\left[ (0.5)^n - (-2)^n \right] u[n] \xleftarrow{\mathcal{Z}} X(z) = \frac{z}{z - 0.5} - \frac{z}{z + 2}, \quad |z| > 2$$
Inverse $z$ Transform Example

Find the inverse $z$ transform of

$$X(z) = \frac{z}{z - 0.5} - \frac{z}{z + 2}, \quad |z| < 0.5$$

In this case, both signals are left sided. Then using

$$-\alpha^n u[-n-1] \xrightarrow{\mathcal{Z}} \frac{z}{z - \alpha} = \frac{1}{1 - \alpha z^{-1}}, \quad |z| < |\alpha|$$

We get

$$-\left[(0.5)^n - (-2)^n\right] u[-n-1] \xrightarrow{\mathcal{Z}} X(z) = \frac{z}{z - 0.5} - \frac{z}{z + 2}, \quad |z| < 0.5$$
The Unilateral $z$ Transform

Just as it was convenient to define a unilateral Laplace transform it is convenient for analogous reasons to define a unilateral $z$ transform

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$
Properties of the Unilateral \( z \) Transform

If two causal discrete-time signals form these transform pairs,
\[
g[n] \xleftarrow{\mathcal{Z}} G(z) \quad \text{and} \quad h[n] \xleftarrow{\mathcal{Z}} H(z)
\]
then the following properties hold for the unilateral \( z \) transform.

Time Shifting

**Delay:** \[
g[n - n_0] \xleftarrow{\mathcal{Z}} z^{-n_0} G(z), \quad n_0 \geq 0
\]

**Advance:** \[
g[n + n_0] \xleftarrow{\mathcal{Z}} z^{n_0} \left( G(z) - \sum_{m=0}^{n_0-1} g[m] z^{-m} \right), \quad n_0 > 0
\]

**Accumulation:**
\[
\sum_{m=0}^{n} g[m] \xleftarrow{\mathcal{Z}} \frac{z}{z-1} G(z)
\]
Solving Difference Equations

The unilateral $z$ transform is well suited to solving difference equations with initial conditions. For example,

$$y[n+2] - \frac{3}{2}y[n+1] + \frac{1}{2}y[n] = (1/4)^n, \quad \text{for } n \geq 0$$

$$y[0] = 10 \quad \text{and} \quad y[1] = 4$$

$z$ transforming both sides,

$$z^2 \left[ Y(z) - y[0] - z^{-1} y[1] \right] - \frac{3}{2}z \left[ Y(z) - y[0] \right] + \frac{1}{2}Y(z) = \frac{z}{z - 1/4}$$

the initial conditions are called for systematically.
Solving Difference Equations

Applying initial conditions and solving,

\[ Y(z) = z \left( \frac{16}{z-1} \right)^{\frac{3}{4}} + \frac{4}{z-1} + \frac{2}{3} \]

and

\[ y[n] = \left[ \frac{16}{3} \left( \frac{1}{4} \right)^n + 4 \left( \frac{1}{2} \right)^n + \frac{2}{3} \right] u[n] \]

This solution satisfies the difference equation and the initial conditions.
Pole-Zero Diagrams and Frequency Response

For a stable system, the response to a sinusoid applied at time $t = 0$ approaches the response to a true sinusoid (applied for all time).

![Diagram of pole-zero diagrams and frequency response]

Response to a Sinusoid

Response to a Suddenly-Applied Sinusoid
Homework 4. Due date: 09/23/2016

20. Draw a Direct Form II block diagram for each of these system transfer functions.

(a) \[ H(z) = \frac{z^2}{2z^4 + 1.2z^3 - 1.06z^2 + 0.08z - 0.02} \]

(b) \[ H(z) = \frac{z^2(z^2 + 0.8z + 0.2)}{(2z^2 + 2z + 1)(z^2 + 1.2z + 0.5)} \]

21. Find the region of convergence in the z plane (if it exists) of the z transform of these signals.

(a) \[ x[n] = (1/2)^n u[n] \]

(b) \[ x[n] = (5/4)^n u[n] + (10/7)^n u[-n] \]
25. A digital filter has an impulse response \( h[n] = \frac{\delta[n] + \delta[n-1] + \delta[n-2]}{10} \).

(a) How many finite poles and finite zeros are there in its transfer function and what are their numerical locations?

(b) If the excitation \( x[n] \) of this system is a unit sequence, what is the final numerical value of the response \( \lim_{n \to \infty} y[n] \)?

26. The forward \( z \) transform \( h[n] = (4/5)u[n] \ast u[n] \) can be expressed in the general form \( H(z) = \frac{b_0 z^2 + b_1 z + b_2}{a_0 z^2 + a_1 z + a_2} \). Find the numerical values of \( b_0, b_1, b_2, a_0, a_1 \) and \( a_2 \).

27. Find the inverse \( z \) transforms of these functions in closed form using partial-fraction expansion, a \( z \)-transform table and the properties of the \( z \) transform.

(a) \( X(z) = \frac{z - 1}{z^2 + 1.8z + 0.82}, \quad |z| > 0.9055 \)

(b) \( X(z) = \frac{z - 1}{z(z^2 + 1.8z + 0.82)}, \quad |z| > 0.9055 \)

(c) \( X(z) = \frac{z^2}{z^2 - z + 1/4}, \quad |z| < 0.5 \)

(d) \( X(z) = \frac{z + 0.3}{z^2 + 0.8z + 0.16}, \quad |z| > 0.4 \)

(e) \( X(z) = \frac{z^2 - 0.8z + 0.3}{z^3}, \quad |z| > 0 \)

28. A signal \( y[n] \) is related to another signal \( x[n] \) by

\[
y[n] = \sum_{m=-\infty}^{n} x[m].
\]

If \( y[n] \xrightarrow{z} \frac{1}{(z-1)^2}, \quad |z| > 1 \), what are the numerical values of \( x[-1], x[0], x[1] \) and \( x[2] \)?
Pole-Zero Diagrams and Frequency Response

Let the transfer function of a system be

\[ H(z) = \frac{z}{z^2 - z/2 + 5/16} = \frac{z}{(z - p_1)(z - p_2)} \]

\[ p_1 = \frac{1 + j2}{4}, \quad p_2 = \frac{1 - j2}{4} \]

\[ |H(e^{j\Omega})| = \left| \frac{e^{j\Omega}}{e^{j\Omega} - p_1} \right| \left| \frac{e^{j\Omega}}{e^{j\Omega} - p_2} \right| \]

\[ z_0 = e^{j\Omega_0} \]
Pole-Zero Diagrams and Frequency Response

| \|H(e^{j\Omega})||  |
|------------------------|
| Closest Approach to a Pole |
| Closest Approach to a Pole |

\[ \|H(e^{j\Omega})\| \]

\[ \angle H(e^{j\Omega}) \]

\[ \Omega \]

\[ -2\pi \to 2\pi \]

\[ -\pi \to \pi \]
A system with transfer function \( H(z) = \frac{z}{(z-0.3)(z+0.8)} \), \( |z| > 0.8 \) is excited by a unit sequence. Find the total response.

Using \( z \)-transform methods,

\[
Y(z) = H(z) X(z) = \frac{z}{(z-0.3)(z+0.8)} \times \frac{z}{z-1} , \quad |z| > 1
\]

\[
Y(z) = \frac{z^2}{(z-0.3)(z+0.8)(z-1)} = -\frac{0.1169}{z-0.3} + \frac{0.3232}{z+0.8} + \frac{0.7937}{z-1} , \quad |z| > 1
\]

\[
y[n] = \left[ -0.1169(0.3)^{n-1} + 0.3232(-0.8)^{n-1} + 0.7937 \right] u[n-1]
\]
Transform Method Comparison

Using the DTFT,

\[ H(e^{j\Omega}) = \frac{e^{j\Omega}}{(e^{j\Omega} - 0.3)(e^{j\Omega} + 0.8)} \]

\[ Y(e^{j\Omega}) = H(e^{j\Omega})X(e^{j\Omega}) = \frac{e^{j\Omega}}{(e^{j\Omega} - 0.3)(e^{j\Omega} + 0.8)} \times \left( \frac{1}{1 - e^{-j\Omega}} + \pi \delta_{2\pi}(\Omega) \right) \]

\[ Y(e^{j\Omega}) = \frac{e^{j2\Omega}}{(e^{j\Omega} - 0.3)(e^{j\Omega} + 0.8)(e^{j\Omega} - 1)} + \pi \frac{e^{j\Omega}}{(e^{j\Omega} - 0.3)(e^{j\Omega} + 0.8)} \delta_{2\pi}(\Omega) \]

\[ Y(e^{j\Omega}) = \frac{-0.1169}{e^{j\Omega} - 0.3} + \frac{0.3232}{e^{j\Omega} + 0.8} + \frac{0.7937}{e^{j\Omega} - 1} + \frac{\pi}{(1 - 0.3)(1 + 0.8)} \delta_{2\pi}(\Omega) \]
Transform Method Comparison

Using the equivalence property of the impulse and the periodicity of both $\delta_{2\pi}(\Omega)$ and $e^{j\Omega}$

$$Y(e^{j\Omega}) = \frac{-0.1169 e^{-j\Omega}}{1-0.3e^{-j\Omega}} + \frac{0.3232 e^{-j\Omega}}{1+0.8e^{-j\Omega}} + \frac{0.7937 e^{-j\Omega}}{1-e^{-j\Omega}} + 2.4933 \delta_{2\pi}(\Omega)$$

Then, manipulating this expression into a form for which the inverse DTFT is direct

$$Y(e^{j\Omega}) = \frac{-0.1169 e^{-j\Omega}}{1-0.3e^{-j\Omega}} + \frac{0.3232 e^{-j\Omega}}{1+0.8e^{-j\Omega}} + 0.7937 \left( \frac{e^{-j\Omega}}{1-e^{-j\Omega}} + \pi \delta_{2\pi}(\Omega) \right)$$

$$\underbrace{-0.7937 \pi \delta_{2\pi}(\Omega)}_{=0} + 2.4933 \delta_{2\pi}(\Omega)$$
Transform Method Comparison

\[
Y(e^{j\Omega}) = \frac{-0.1169e^{-j\Omega}}{1 - 0.3e^{-j\Omega}} + \frac{0.3232e^{-j\Omega}}{1 + 0.8e^{-j\Omega}} + 0.7937 \left( \frac{e^{-j\Omega}}{1 - e^{-j\Omega}} + \pi \delta_{2\pi}(\Omega) \right)
\]

Finding the inverse DTFT,

\[
y[n] = \left[ -0.1169(0.3)^{n-1} + 0.3232(-0.8)^{n-1} + 0.7937 \right] u[n-1]
\]

The result is the same as the result using the z transform, but the effort and the probability of error are considerably greater.
System Response to a Sinusoid

A system with transfer function

\[ H(z) = \frac{z}{z - 0.9} , \quad |z| > 0.9 \]

is excited by the sinusoid \( x[n] = \cos(2\pi n / 12) \). Find the response.

The \( z \) transform of a true sinusoid does not appear in the table of \( z \) transforms. The \( z \) transform of a causal sinusoid of the form \( x[n] = \cos(2\pi n / 12)u[n] \) does appear. We can use the DTFT to find the response to the true sinusoid and the result is \( y[n] = 1.995 \cos(2\pi n / 12 - 1.115) \).
System Response to a Sinusoid

Using the $z$ transform we can find the response of the system to a causal sinusoid $x[n] = \cos(2\pi n / 12)u[n]$ and the response is

$$y[n] = 0.1217(0.9)^n u[n] + 1.995 \cos(2\pi n / 12 - 1.115)u[n]$$

Notice that the response consists of two parts, a transient response $0.1217(0.9)^n u[n]$ and a forced response $1.995 \cos(2\pi n / 12 - 1.115)u[n]$ that, except for the unit sequence factor, is exactly the same as the forced response we found using the DTFT.
System Response to a Sinusoid

This type of analysis is very common. We can generalize it to say that if a system has a transfer function \( H(z) = \frac{N(z)}{D(z)} \) that the response to a causal cosine excitation \( \cos(\Omega_0 n) u[n] \) is

\[
y[n] = \mathcal{Z}^{-1} \left( \frac{N(z)}{D(z)} \right) + \left| H(p_1) \right| \cos(\Omega_0 n + \angle H(p_1)) u[n]
\]

where \( p_1 = e^{j\Omega_0} \). This consists of a natural or transient response and a forced response. If the system is stable the transient response dies away with time leaving only the forced response which, except for the \( u[n] \) factor is the same as the forced response to a true cosine. So we can use the \( z \) transform to find the response to a true sinusoid.