Today:

– Linear Programming (con’t.)

COSC 581, Algorithms
April 3, 2014
Reading Assignments

• Today’s class:
  – Chapter 29.2

• Reading assignment for next Thursday’s class:
  – Chapter 29.3-4
First, a bit of review...
The General LP Problem

maximize  \( c_1 x_1 + c_2 x_2 + \cdots + c_d x_d \)  
subject to:  
\[
\begin{align*}
  a_{11} x_1 + a_{12} x_2 + \cdots + a_{1d} x_d & \leq b_1 \\
  a_{21} x_1 + a_{22} x_2 + \cdots + a_{2d} x_d & \leq b_2 \\
  \vdots \\
  a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nd} x_d & \leq b_n
\end{align*}
\]
General Steps of LP

Step 1: Determine the decision variables
Step 2: Determine the objective function
Step 3: Determine the constraints
Step 4: Convert into standard or slack form
Step 5: Solve
Two Canonical Forms for LP: Standard and Slack

- An LP is in **standard form** if it is the **maximization** of a linear function subject to linear inequalities.

- An LP is in **slack form** if it is the **maximization** of a linear function subject to linear equalities.
Equivalence of Linear Programs

- Two maximization LPs, $L$ and $L'$, are equivalent if for each feasible solution $x$ to $L$ with objective value $z$ there is a corresponding feasible solution $x'$ to $L'$ with objective value $z$, and vice versa.

- A maximization LP, $L$, and a minimization LP, $L'$, are equivalent if for each feasible solution $x$ to $L$ with objective value $z$ there is a corresponding feasible solution $x'$ to $L'$ with objective value $-z$, and vice versa.
Standard Form

• We’re given:
  
  \( n \) real numbers \( c_1, c_2, \ldots, c_n \)
  
  \( m \) real numbers \( b_1, b_2, \ldots, b_m \)
  
  \( mn \) real numbers \( a_{ij} \), for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \)

• We want to find:
  
  \( n \) real numbers \( x_1, x_2, \ldots, x_n \) that:

  Maximize: \( \sum_{j=1}^{n} c_j x_j \)

  Subject to:

  \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \ldots, m \]

  \[ x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n \]
Compact Version of Standard Form

• Let: $A = (a_{ij})$ be $m \times n$ matrix
  $b = (b_i)$ be an $m$-vector
  $c = (c_j)$ be an $n$-vector
  $x = (x_j)$ be an $n$-vector

• Rewrite LP as:
  Maximize: $c^T x$
  Subject to:
  $Ax \leq b$
  $x \geq 0$

• Now, we can concisely specify LP in standard form as $(A, b, c)$
Slack Form – Useful for Simplex

- In slack form, the only inequality constraints are the non-negativity constraints
  - All other constraints are equality constraints
- Let:
  \[ \sum_{i=1}^{n} a_{ij} x_j \leq b_i \]
  be an inequality constraint
- Introduce new variable \( s \), and rewrite as:
  \[ s = b_i - \sum_{j=1}^{n} a_{ij} x_j \]
  \[ s \geq 0 \]
- \( s \) is a slack variable; it represents difference between left-hand and right-hand sides
Slack Form (con’t.)

• In general, we’ll use $x_{n+i}$ (instead of $s$) to denote the slack variable associated with the $i$th inequality.

• The $i$th constraint is therefore:

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_i$$

along with the non-negativity constraint $x_{n+i} \geq 0$
Example

Standard form:
Maximize $2x_1 - 3x_2 + 3x_3$
subject to:
\begin{align*}
    x_1 + x_2 - x_3 & \leq 7 \\
    -x_1 - x_2 + x_3 & \leq -7 \\
    x_1 - 2x_2 + 2x_3 & \leq 4 \\
    x_1, x_2, x_3 & \geq 0
\end{align*}

Slack form:
Maximize $2x_1 - 3x_2 + 3x_3$
subject to:
\begin{align*}
    x_4 &= 7 - x_1 - x_2 + x_3 \\
    x_5 &= -7 + x_1 + x_2 - x_3 \\
    x_6 &= 4 - x_1 + 2x_2 - 2x_3 \\
    x_1, x_2, x_3, x_4, x_5, x_6 & \geq 0
\end{align*}
Concise Representation of Slack Form

- Can eliminate “maximize”, “subject to”, and non-negativity constraints (all are implicit)
- And, introduce $z$ as value of objective function:
  
  $z = 2x_1 - 3x_2 + 3x_3$
  $x_4 = 7 - x_1 - x_2 + x_3$
  $x_5 = -7 + x_1 + x_2 - x_3$
  $x_6 = 4 - x_1 + 2x_2 - 2x_3$

- Then, define slack form of LP as tuple $(N, B, A, b, c, v)$
  where $N =$ indices of nonbasic variables
  $B =$ indices of basic variables

- We can rewrite LP as:

  $z = v + \sum_{j \in N} c_j x_j$

  $x_i = b_i - \sum_{j \in N} a_{ij} x_j$ for $i \in B$
Setting problems as LPs

- **Single Source Shortest Path:**
  - Input: A weighted directed graph $G=<V,E>$ with weighted function $w: E \rightarrow \mathbb{R}$, a source $s$ and a destination $t$, compute $d$ which is the weight of the shortest path from $s$ to $t$.
  - Formulate as a LP:
    - For each vertex $v$, introduce a variable $d_v$: the weight of the shortest path from $s$ to $v$.
    - LP:
      
      \[
      \begin{align*}
      \text{maximize} & \quad d_t \\
      \text{subject to:} & \quad d_v \leq d_u + w(u,v) \quad \text{for each edge } (u,v) \in E \\
      & \quad d_s = 0
      \end{align*}
      \]

- Q: Why is this a maximization?
- Q: How many variables?
- Q: How many constraints?
Formatting problems as LPs

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Q: How many constraints?
Formatting problems as LPs – SSSP

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      & & \quad d_s = 0
      \end{align*}
      \]

Q: Why is this a maximization?
Q: How many variables? $|V|$
Q: How many constraints? $|E|+1$
Formatting problems as LPs – Max Flow

• Recall (how could you forget?) Max-flow problem:
  – A directed graph $G=\langle V,E \rangle$, a capacity function on each edge $c(u,v) \geq 0$ and a source $s$ and a sink $t$. A flow is a function $f : V \times V \to \mathbb{R}$ that satisfies:
    • Capacity constraints: for all $u,v \in V$, $f(u,v) \leq c(u,v)$.
    • Skew symmetry: for all $u,v \in V$, $f(u,v) = -f(v,u)$.
    • Flow conservation: for all $u \in V - \{s,t\}$, $\sum_{v \in V} f(u,v) = 0$
  – Find a maximum flow from $s$ to $t$. 
Formatting Max-flow problem as LP

maximize $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$

subject to:

$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}$ for all $u \in V - \{s,t\}$ //flow conservation

$\sum_{v \in V} f_{uv} \leq c(u,v)$ for all $u, v \in V$ //capacity constraints

$f_{uv} \geq 0$ for all $u, v \in V$ //non-negativity constraints

Q: How many variables?
Q: How many constraints?
maximize $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$

subject to:

$f_{uv} \leq c(u,v)$ \quad \text{for all } u, v \in V \quad //\text{capacity constraints}

$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}$ \quad \text{for all } u \in V - \{s,t\} \quad //\text{flow conservation}

$f_{uv} \geq 0$ \quad \text{for all } u, v \in V \quad //\text{non-negativity constraints}

Q: How many variables? $|V|^2$

Q: How many constraints?
maximize $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$

subject to:

$f_{uv} \leq c(u,v)$ for all $u,v \in V$ //capacity constraints

$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}$ for all $u \in V - \{s,t\}$ //flow conservation

$f_{uv} \geq 0$ for all $u,v \in V$ //non-negativity constraints

Q: How many variables? $|V|^2$

Q: How many constraints? $2|V|^2 + |V| - 2$
Lots of “standard” problems can be formulated as LPs

• Question:

When do you use specialized algorithms (like Dijkstra for SSSP), and when do you use LP (like the LP formulation we just made for SSSP)?
Lots of “standard” problems can be formulated as LPs

• Question:
  When do you use specialized algorithms (like Dijkstra for SSSP), and when do you use LP (like the LP formulation we just made for SSSP)?

• Answer:
  – Specialized solutions often provide better runtime performance
  – But, when specialized solutions aren’t available, LP gives a “generic” approach applicable to many types of problems
The Simplex algorithm for LP

- Classical method for solving LP problems
- Very simple
- Worst case run time is not polynomial
- But, often very fast in practice
Recall Important Observation:
Optimal Solutions are at a Vertex or Line Segment

- Intersection of objective function and feasible region is either vertex or line segment
- Feasible region is **convex** – makes optimization much easier!
- **Simplex algorithm** finds LP solution by:
  - Starting at some vertex
  - Moving along edge of simplex to neighbor vertex whose value is at least as large
  - Terminates when it finds local maximum
- **Convexity ensures this local maximum is globally optimal**
Example for Simplex algorithm

Maximize $3x_1 + x_2 + 2x_3$

Subject to:
\[
\begin{align*}
    x_1 + x_2 + 3x_3 & \leq 30 \\
    2x_1 + 2x_2 + 5x_3 & \leq 24 \\
    4x_1 + x_2 + 2x_3 & \leq 36 \\
    x_1, x_2, x_3 & \geq 0
\end{align*}
\]

Change to slack form:
\[
\begin{align*}
    z &= 3x_1 + x_2 + 2x_3 \\
    x_4 &= 30 - x_1 - x_2 - 3x_3 \\
    x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
    x_6 &= 36 - 4x_1 - x_2 - 2x_3 \\
    x_1, x_2, x_3, x_4, x_5, x_6 & \geq 0
\end{align*}
\]
Recall, regarding Slack Form...

Slack form:

Maximize \(2x_1 - 3x_2 + 3x_3\)

subject to:

\[
\begin{align*}
  x_4 &= 7 - x_1 - x_2 + x_3 \\
  x_5 &= -7 + x_1 + x_2 - x_3 \\
  x_6 &= 4 - x_1 + 2x_2 - 2x_3 \\
  x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{align*}
\]
Simplex algorithm steps

• Recall: “Feasible solutions” (infinite number of them):
  – A feasible solution is any whose values satisfy constraints
  – In previous example, solution is feasible as long as all of $x_1, x_2, x_3, x_4, x_5, x_6$ are nonnegative

• Basic solution:
  – set all nonbasic variables to 0 and compute all basic variable values

• Iteratively rewrite the set of equations such that:
  – There is no change to the underlying LP problem (i.e., new form is equivalent to old)
  – Feasible solutions stay the same
  – The basic solution is changed, to result in a greater objective value:
    • Select a nonbasic variable $x_e$ whose coefficient in the objective function is positive
    • Increase value of $x_e$ as much as possible without violating any of the constraints
    • Make $x_e$ a basic variable
    • Select some other variable to become nonbasic

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$
Example

- **Basic solution:** \((x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 30, 24, 36)\)
  - The objective value is \(z = 3 \cdot 0 + 0 + 2 \cdot 0 = 0\) (Not a maximum)

- Try to increase the value of **nonbasic variable** \(x_1\) while maintaining constraints:
  
  Increase \(x_1\) to 30: means that \(x_4\) will be OK (i.e., non-negative)
  
  Increase \(x_1\) to 12 means that \(x_5\) will be OK 9:
  
  Increase \(x_1\) to 9 means that \(x_6\) will be OK.
  
  We have to choose most constraining value \(\Rightarrow x_1\) is most constrained by \(x_6\), so we switch the roles of \(x_1\) and \(x_6\)

- Change \(x_1\) to **basic** variable by rewriting last constraint to:
  
  \(x_1 = 9 - x_2/4 - x_3/2 - x_6/4\)
  
  - Note: \(x_6\) becomes nonbasic.
  
  - Replace \(x_1\) with above formula in all equations to get...
Example (con’t.)

\[ z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \]
\[ x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \]
\[ x_4 = 21 - 3\frac{x_2}{4} - 5\frac{x_3}{2} + \frac{x_6}{4} \]
\[ x_5 = 6 - 3\frac{x_2}{2} - 4x_3 + \frac{x_6}{2} \]

• This operation is called **pivot**
  – A pivot chooses a nonbasic variable, called **entering variable**, and a basic variable, called **leaving variable**, and changes their roles.
  – The pivot operation results in an equivalent LP.
  – Reality check: original solution \((0,0,0,30,24,36)\) satisfies the new equations.

• In the example,
  – \(x_1\) is entering variable, and \(x_6\) is leaving variable.
  – \(x_2, x_3, x_6\) are nonbasic, and \(x_1, x_4, x_5\) becomes basic.
  – The basic solution for this new LP form is \((9,0,0,21,6,0)\), with \(z = 27\).
    \((Yippee \rightarrow z = 27 \text{ is better than } z = 0!)\)
Example (con’t.)

• We iterate again –try to find a new variable whose value may increase.
  – $x_6$ will not work, since $z$ will decrease.
  – $x_2$ and $x_3$ are OK. Suppose we select $x_3$.

• How far can we increase $x_3$?
  – First constraint limits it to 18
  – Second constraint limits it to $42/5$
  – Third constraint limits it to $3/2$ – most constraining $\Rightarrow$ swap roles of $x_3$ and $x_5$

• So rewrite last constraint to:
  $$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{2}x_6$$

• Replace $x_3$ with the above in all the equations to get...

\[
\begin{align*}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3}{4}x_6 \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{1}{4}x_6 \\
x_4 &= 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6 \\
x_5 &= 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6
\end{align*}
\]
Example (con’t.)

• The new LP equations:
  – \( z=111/4+x_2/16 -x_5/8 - 11x_6/16 \)
  – \( x_1=33/2- x_2/16 +x_5/8 - 5x_6/16 \)
  – \( x_3=3/2-3x_2/8 -x_5/4+x_6/8 \)
  – \( x_4=69/4+3x_2/16 +5x_5/8-x_6/16 \)
• The basic solution is \((33/4,0,3/2,69/4,0,0)\) with \(z=111/4\).

• Now we can only increase \(x_2\).
  – First constraint limits \(x_2\) to 132
  – Second to 4
  – Third to \(\infty\)
• So rewrite second constraint to:
  \( x_2 = 4 - 8x_3/3 - 2x_5/3 + x_6/3 \)
• Replace in all equations to get...
Example (con’t.)

- Rewritten LP equations:
  \[ z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - 2\frac{x_6}{3} \]
  \[ x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \]
  \[ x_2 = 4 - 8\frac{x_3}{3} - 2\frac{x_5}{3} + \frac{x_6}{3} \]
  \[ x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \]
- At this point, all coefficients in objective functions are negative.
- So, no further rewrite is possible.
- Means that we’ve found the optimal solution.
- The basic solution is \((8, 4, 0, 18, 0, 0)\) with objective value \(z = 28\).
- The original variables are \(x_1, x_2, x_3\), with values \((8, 4, 0)\)
Next time...

• More details on the correctness and optimality of SIMPLEX
Reading Assignments

• Reading assignment for next Thursday’s class:
  – Chapter 29.3-4