Karnaugh Maps & Sum-of-Products

Representation Forms [1]

- **Sum-of-products form**
  - Boolean expression consisting of a sum of terms where each term is a “product” containing exactly one instance of every variable, e.g.,

  \[ F_1(A, B, C) = \overline{A}BC + \overline{A}B\overline{C} + AB\overline{C} \]

Representation Forms [2]

- **Product-of-sums form**
  - Boolean expression consisting of a product of terms where each term is a “sum” containing exactly one instance of every variable, e.g.,

  \[ F_2(A, B, C) = (\overline{A} + \overline{B} + C) (A + \overline{B} + \overline{C}) (A + \overline{B} + C) \]

  \[ F_1(A, B, C) = F_2(A, B, C) \quad \text{(Verify yourself)} \]

### Sum-of-Products

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>F</th>
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Can use to express a function using variables that yield a function value of 1.

\[ F = \overline{A}BC + \overline{A}B\overline{C} + AB\overline{C} \]

\[ m(1, 2, 4) \]

\[ \sum \text{m}(1, 2, 4) \]
### Product-of-Sums

Can use to express a function using variables that yield a function value of 0.

$F = (A+B+C)(\overline{A}+\overline{B}+\overline{C})$

$= \prod M(0, 7)$

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### Karnaugh Maps [1]

- Invented by Maurice Karnaugh
- Logic expressions quickly become complex to read and understand
- Simplification is not easy
  - We have to remember all of the laws we can apply to an expression
- Karnaugh maps provide easy mechanism to simplify expressions
  - Uses a graphical method
  - Does NOT cover all laws

### Karnaugh Maps [2]

- Feasible for up to 4-6 variables (harder to display graphically beyond 4)
- Map is an array of $2^n$ cells, representing the possible combinations of the values of $n$ binary variables
- Karnaugh map is derived from a truth table (sop form of function)

### 2-Variable K-Map [1]

- Simple example...
- One cell in the map for each row in the truth table
- e.g.- Expression: $F(X, Y) = \overline{X}Y + \overline{X}Y + XY$

**Truth Table:**

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>F</th>
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<tbody>
<tr>
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**K-Map:**

$XY$

$00 \ 01 \ 11 \ 10$

$1 \ 1 \ 1$

Note the order of the list of values for $XY$ – adjacent cells change the value of only one variable.
2-Variable K-Map

K-Map:

Group adjacent cells with 1's into power-of-two sized groupings. Include all 1's. May re-use a cell. Each group of more than 1 cell represents a simplification.

\[ m_1 + m_3 = \overline{X}Y + XY = (\overline{X} + X)Y = Y \]

\[ m_3 + m_2 = XY + \overline{X}Y = X(Y + \overline{Y}) = X \]

Thus, \( F(X,Y) = X + Y \)

3-Variable K-Map

\[ F(X,Y,Z) = XYZ + XYZ + XYZ + XYZ \]

\[ m_0 + m_1 = XYZ + XYZ = XY \]

\[ m_3 + m_7 = XYZ + XYZ = YZ \]

\[ F(X,Y,Z) = XY + YZ \]

Example 1: \( F(A,B,C) = AB + C \)

Example 2: \( F(A,B,C) = AC + AB + ABC + BC \)

Note that function is not in SOP form, can expand. For example, \( AC \) expands to \( ABC + ABC \).

\[ m_1 + m_3 + m_5 + m_7 = ABC + ABC + ABC + ABC \]

\[ = \overline{A}C + AC = C \]

\[ m_1 + m_2 = \overline{A}BC + ABC \]

\[ = \overline{A}B \]

\[ F(A,B,C) = \overline{A}B + C \]

4-Variable K-Map

Example 1: \( F(W,X,Y,Z) = \Sigma m(0,2,5,8,10,13) \)

\[ m_0 + m_2 + m_8 + m_{10} = \overline{W}XYZ + \overline{W}XYZ + WXY \overline{Z} + WXYZ \]

\[ = XZ \]

\[ m_5 + m_{13} = \overline{W}XYZ + WXYZ = XYZ \]

\[ F(W,X,Y,Z) = \overline{X}Y + XYZ \]
4-Variable K-Map [2]

- Example 2: \( F(W,X,Y,Z) = \sum m(0,1,2,4,6,14,15) \)

\[
\begin{array}{cccc}
00 & 01 & 11 & 10 \\
01 & 1 & 1 & 1 \\
11 & 1 & 1 & 0 \\
10 & & & \\
\end{array}
\]

\[
m_0 + m_2 + m_4 + m_6 = \overline{W}XYZ + \overline{W}XYZ + \overline{W}XYZ + \overline{W}XYZ = \overline{W}Z
\]

\[
m_{14} + m_{15} = WXYZ + WXYZ = WXY
\]

\[
m_0 + m_1 = \overline{W}XYZ + \overline{W}XYZ = \overline{W}XY
\]

\[F(W,X,Y,Z) = \overline{W}Z + \overline{W}XY + WXY\]

Don’t Care Conditions

\[F = \sum m(1,3,7,11,15)\]

\[x = \sum m(0,5)\]

ABCD & ABCD never occur; thus, don’t care what value is associated with them. Use those cells to your advantage (can ignore or use as 1).

\[
\begin{array}{cccc}
00 & 01 & 11 & 10 \\
01 & x & 1 & 1 \\
11 & x & 1 & \\
10 & 1 & 1 & \\
\end{array}
\]

\[F = (\overline{A} + C)D\]

5-Variable K-Map

6-Variable K-Map