Mathematical Description of Continuous-Time Signals
Typical Continuous-Time Signals

![Amplitude-Modulated Carrier in a Communication System](image1)

![Step Response of an RC Lowpass Filter](image2)

![Car Bumper Height After Car Strikes a Speed Bump](image3)

![Light Intensity from a Q-Switched Laser](image4)

![Frequency-Shift-Keyed Binary Bit Stream](image5)

![Manchester Encoded Baseband Binary Bit Stream](image6)
Continuous vs Continuous-Time Signals

All continuous signals that are functions of time are continuous-time, but not all continuous-time signals are continuous. All these signals are continuous-time. a, b and c are continuous but d is not.
Continuous-Time Sinusoids

\[ g(t) = A \cos \left( \frac{2\pi t}{T_0} + \theta \right) = A \cos \left( 2\pi f_0 t + \theta \right) = A \cos \left( \omega_0 t + \theta \right) \]

- Amplitude
- Period (s)
- Phase Shift (radians)
- Cyclic Frequency (Hz)
- Radian Frequency (radians/s)

\[ \omega_0 = 2\pi f_0 \]

\[ g(t) = A \cos \left( 2\pi f_0 t + \theta \right) \]
Continuous-Time Exponentials

\[ g(t) = Ae^{-t/\tau} \]

Amplitude, Time Constant (s)

\[ g(t) \]

[Diagram showing the function \( g(t) = Ae^{-t/\tau} \) with amplitude \( A \) and time constant \( \tau \).]
Complex Sinusoids

Euler's Identity: $e^{jx} = \cos(x) + j\sin(x)$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}, \quad \sin(x) = \frac{e^{jx} - e^{-jx}}{j2}$$
The Signum Function

\[ \text{sgn}(t) = \begin{cases} 
1 & , t > 0 \\
0 & , t = 0 \\
-1 & , t < 0 
\end{cases} \]

The signum function, in a sense, returns an indication of the sign of its argument.
The Unit Step Function

\[ u(t) = \begin{cases} 
1 & , \ t > 0 \\
1/2 & , \ t = 0 \\
0 & , \ t < 0
\end{cases} \]

\[ u(t) = \frac{1}{2} [ \text{sgn}(t) + 1 ] \]

The product signal \( g(t)u(t) \) can be thought of as the signal \( g(t) \) “turned on” at time \( t = 0 \).
The Unit Step Function

The unit step function can mathematically describe a signal that is zero up to some point in time and non-zero after that.

\[
\begin{align*}
V_b & \quad t = 0 \\
V_{RC}(t) & \quad v_C(t) \\
i_{RC}(t) & \quad C \\
\end{align*}
\]

\[
\begin{align*}
v_{RC}(t) &= V_b u(t) \\
i_{RC}(t) &= \left(\frac{V_b}{R}\right)e^{-t/RC} u(t) \\
v_C(t) &= V_b \left(1 - e^{-t/RC}\right) u(t)
\end{align*}
\]
The Unit Ramp Function

\[ \text{ramp}(t) = \begin{cases} t & , \quad t > 0 \\ 0 & , \quad t \leq 0 \end{cases} = \int_{-\infty}^{t} u(\lambda) \, d\lambda = tu(t) \]
The Unit Ramp Function

Product of a sine wave and a ramp function.

\[ x(t) \]
Introduction to the Impulse

Define a function \( \Delta(t) = \begin{cases} \frac{1}{a}, & |t| < a/2 \\ 0, & |t| > a/2 \end{cases} \)

Let another function \( g(t) \) be finite and continuous at \( t = 0 \).
Introduction to the Impulse

The Sampling Effect of the Unit Impulse

The area under the product of the two functions is

\[ A = \frac{1}{a} \int_{-a/2}^{a/2} g(t) dt = \frac{1}{a} \int_{-a/2}^{a/2} g(t) dt \]

As the width of \( \Delta(t) \) approaches zero, \( \lim_{a \to 0} \Delta(t) = \delta(t) \) and

\[ \lim_{a \to 0} A = g(0) \lim_{a \to 0} \frac{1}{a} \int_{-a/2}^{a/2} dt = g(0) \lim_{a \to 0} \frac{1}{a} (\alpha) = g(0) \]

The continuous-time unit impulse is implicitly defined by

\[ g(0) = \int_{-\infty}^{\infty} \delta(t) g(t) dt \]
Introduction to the Impulse

Alternate Derivation of the Sampling Effect of the Unit Impulse

Since \( g(t) \) is continuous at \( t = 0 \), it can be represented by a Maclaurin series as

\[
g(t) = g(0) + \frac{g'(0)}{1!} t + \frac{g''(0)}{2!} t^2 + \cdots
\]

\[
A = \frac{1}{a} \int_{-a/2}^{a/2} g(t) \, dt = \frac{1}{a} \int_{-a/2}^{a/2} \left[ g(0) + g'(0) t + \frac{g''(0)}{2} t^2 + \cdots \right] \, dt
\]

As the width of \( \Delta(t) \) approaches zero,

\[
\lim_{a \to 0} A = \lim_{a \to 0} \left[ \frac{1}{a} \int_{-a/2}^{a/2} g(0) \, dt + \frac{1}{a} \int_{-a/2}^{a/2} g'(0) \, dt + \frac{1}{2a} \int_{-a/2}^{a/2} g''(0) \, dt + \cdots \right]
\]

\[
\lim_{a \to 0} A = \lim_{a \to 0} \left[ \frac{1}{a} g(0) a + \frac{1}{a} g'(0)(0) + \frac{1}{a} g''(0) \left( \frac{a^3}{4} \right) + \cdots \right] = g(0)
\]
The Unit Step and Unit Impulse

As \( \alpha \) approaches zero, \( g(t) \) approaches a unit step and \( g'(t) \) approaches a unit impulse.

The unit step is the integral of the unit impulse and the unit impulse is the generalized derivative of the unit step.
Graphical Representation of the Impulse

The impulse is not a function in the ordinary sense because its value at the time of its occurrence is not defined. It is represented graphically by a vertical arrow. Its strength is either written beside it or is represented by its length.

\[
\begin{align*}
\delta(t) & \quad \delta(t) & \quad 9\delta(t-1) & \quad -3\delta(t+2) \\
(1) & \quad 1 & \quad 9 & \quad -2 \\
t & \quad t & \quad t & \quad t
\end{align*}
\]
Properties of the Impulse

The Sampling Property

\[ \int_{-\infty}^{\infty} g(t) \delta(t - t_0) \, dt = g(t_0) \]

The sampling property “extracts” the value of a function at a point.

The Scaling Property

\[ \delta(\alpha(t - t_0)) = \frac{1}{|\alpha|} \delta(t - t_0) \]

This property illustrates that the impulse is different from ordinary mathematical functions.
Properties of the Impulse

\[
\int [3e^{-t/6} \times 5\delta(t-1)] dt = 15e^{-1/6} = 12.6972
\]

\[
\int \left\{9(t^3 + 4) \times [-4\delta(t+2)]\right\} dt = -36\left[(-2)^3 + 4\right] = 144
\]

\[
\int \left[7\sin(200\pi t) \times 95\delta(t-3)\right] dt = 665\sin(600\pi) = 0
\]

\[
10\delta(3(t-2)) = \frac{10}{3}\delta(t-2)
\]

\[
22\delta(-7(t+10)) = \frac{22}{7}\delta(t+10)
\]

\[
13\delta(4t+9) = 13\delta(4(t+9/4)) = \frac{13}{4}\delta(t+9/4)
\]
The Unit Periodic Impulse

The unit periodic impulse is defined by

\[ \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad n \text{ an integer} \]

The periodic impulse is a sum of infinitely many impulses uniformly-spaced apart by \( T \).
The Periodic Impulse

\[ \delta_4(t) \]

\[ -3\delta_{0.25}(t + 0.2) \]

\[ 16\delta_8(t - 3) \]
Properties of the Periodic Impulse

The unit periodic impulse is defined by

\[ \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) , \text{ } n \text{ an integer} \]

Therefore

\[ \delta_T(a(t-t_0)) = \sum_{n=-\infty}^{\infty} \delta(a(t-t_0)-nT) = \sum_{n=-\infty}^{\infty} \delta(a(t-t_0-nT/a)) , \text{ } n \text{ an integer} \]

and, by the scaling property of the impulse,

\[ \delta_T(a(t-t_0)) = \frac{1}{|a|} \sum_{n=-\infty}^{\infty} \delta(t-t_0-nT/a) , \text{ } n \text{ an integer} \]

Then, again using the definition of the periodic impulse,

\[ \delta_T(a(t-t_0)) = \frac{1}{|a|} \delta_{T/a}(t-t_0) , \text{ } n \text{ an integer} \]
The Unit Rectangle Function

\[ \text{rect}(t) = \begin{cases} 
1 & , \quad |t| < 1/2 \\
1/2 & , \quad |t| = 1/2 \\
0 & , \quad |t| > 1/2 
\end{cases} = u(t+1/2) - u(t-1/2) \]

The product signal \( g(t)\text{rect}(t) \) can be thought of as the signal \( g(t) \) “turned on” at time \( t = -1/2 \) and “turned back off” at time \( t = +1/2 \).
Combinations of Functions

\[ [\sin(4\pi t) + 2] \cos(40\pi t) \]

\[ \frac{\sin(4\pi t)}{4\pi t} \]

\[ e^{2t} \cos(10\pi t) \]

\[ \cos(20\pi t) + \cos(22\pi t) \]
Shifting and Scaling Functions

Let a function $g(t)$ be defined by this graph and let $g(t) = 0, \ |t| > 5$

<table>
<thead>
<tr>
<th>$t$</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
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<tr>
<td>$g(t)$</td>
<td>0</td>
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</tbody>
</table>
Shifting and Scaling Functions

Amplitude Scaling, \( g(t) \rightarrow Ag(t) \)
Shifting and Scaling Functions

Time shifting, $t \rightarrow t - t_0$

\[
\begin{array}{cccccccccc}
  t & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
  g(t) & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 \\
\end{array}
\]
Shifting and Scaling Functions

Time scaling, \( t \rightarrow t / a \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>-10</th>
<th>-8</th>
<th>-6</th>
<th>-4</th>
<th>-2</th>
<th>0</th>
<th>2</th>
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</tr>
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<tbody>
<tr>
<td>( g(t) )</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<th>3</th>
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</table>
Shifting and Scaling Functions

Multiple transformations \( g(t) \rightarrow Ag\left(\frac{t-t_0}{a}\right) \)

A multiple transformation can be done in steps

\[
g(t) \xrightarrow{\text{amplitude scaling, } A} Ag(t) \xrightarrow{t \rightarrow t/a} Ag\left(\frac{t}{a}\right) \xrightarrow{t \rightarrow t-t_0} Ag\left(\frac{t-t_0}{a}\right)
\]

The sequence of the steps is significant

\[
g(t) \xrightarrow{\text{amplitude scaling, } A} Ag(t) \xrightarrow{t \rightarrow t_0} Ag\left(t-t_0\right) \xrightarrow{t \rightarrow t/a} Ag\left(\frac{t-t_0}{a}\right) \neq Ag\left(\frac{t-t_0}{a}\right)
\]
Shifting and Scaling Functions

Simultaneous scaling and shifting $g(t) \rightarrow Ag\left(\frac{t-t_0}{a}\right)$
Shifting and Scaling Functions

Simultaneous scaling and shifting, $g(t) \rightarrow Ag(bt - t_0)$
Shifting and Scaling Functions

\[ 3\text{rect}\left(\frac{t+1}{4}\right) \]

\[ 4u(3-t) \]

\[ -5\text{ramp}(0.1t) \]

\[ -3\text{sgn}(2t) \]
Shifting and Scaling Functions

If \( g_2(t) = Ag_1 \left( \frac{(t - t_0)}{w} \right) \) what are \( A, t_0 \) and \( w \)?
Shifting and Scaling Functions

Height +5  →  -2  ⇒  \( A = -0.4 \), \( g_1(t) \rightarrow -0.4g_1(t) \)

Width +6 → +2  ⇒  \( w = 1/3 \)  ⇒  \(-0.4g_1(t) \rightarrow -0.4g_1(3t) \)

Shift left by 5/3  ⇒  \( t_0 = -5/3 \)  ⇒  \(-0.4g_1(3t) \rightarrow -0.4g_1(3(t+5/3)) \)
Shifting and Scaling Functions

If $g_2(t) = Ag_1(wt - t_0)$ what are $A$, $t_0$ and $w$?
Shifting and Scaling Functions

Height $+5 \rightarrow -2 \Rightarrow A = -0.4 \Rightarrow g_1(t) \rightarrow -0.4 g_1(t)$

Shift left $5 \Rightarrow t_0 = -5 \Rightarrow -0.4 g_1(t) \rightarrow -0.4 g_1(t+5)$

Width $+6$ to $+2 \Rightarrow w = 3 \Rightarrow -0.4 g_1(t+5) \rightarrow -0.4 g_1(3t+5)$
Shifting and Scaling Functions

If $g_2(t) = Ag_1(w(t - t_0))$ what are $A$, $t_0$ and $w$?
Shifting and Scaling Functions

Height $+5 \rightarrow -3 \Rightarrow A = -0.6 \Rightarrow g_1(t) \rightarrow -0.6 \cdot g_1(t)$

Width $+6 \rightarrow -3 \Rightarrow w = -2 \Rightarrow -0.6 \cdot g_1(t) \rightarrow -0.6 \cdot g_1(-2t)$

Shift Right $1/2 \Rightarrow t_0 = 1/2 \Rightarrow -0.6 \cdot g_1(-2t) \rightarrow -0.6 \cdot g_1(-2(t-1/2))$
Shifting and Scaling Functions

If \( g_2(t) = Ag_1(t/w - t_0) \) what are \( A, t_0 \) and \( w \)?
Shifting and Scaling Functions

Height $+5 \rightarrow -3 \Rightarrow A = -0.6 \Rightarrow g_1(t) \rightarrow -0.6g_1(t)$

Shift Left $1 \Rightarrow t_0 = -1 \Rightarrow -0.6g_1(t) \rightarrow -0.6g_1(t+1)$

Width $+6 \rightarrow -3 \Rightarrow w = -1/2 \Rightarrow -0.6g_1(t+1) \rightarrow -0.6g_1(-2t+1)$
The horizontal sync pulse can be described by a rectangle function. The color burst can be described by a sinusoid multiplied by a rectangle function.
BPSK Binary Data Signal

BPSK data transmission is a sequence of sinusoidal cycles whose polarities are determined by the data. The data can be described by a sequence of rectangles. The BPSK signal is the data sequence multiplied by a carrier sinusoid.
An EKG waveform can be reasonably approximated by four linear functions in the QRS interval and the P and T pulses can be approximated by Gaussian waveforms.
Differentiation
Integration

\[ \int x(\tau) d\tau \]

\[ \int x(\tau) d\tau \]
Even and Odd Signals

Even Functions
\[ g(t) = g(-t) \]

Odd Functions
\[ g(t) = -g(-t) \]

Even Function
\[ g(t) \]

Odd Function
\[ g(t) \]
Even and Odd Parts of Functions

The **even part** of a function is \( g_e(t) = \frac{g(t) + g(-t)}{2} \).

The **odd part** of a function is \( g_o(t) = \frac{g(t) - g(-t)}{2} \).

A function whose even part is zero is odd and a function whose odd part is zero is even.

The derivative of an even function is odd and the derivative of an odd function is even.

The integral of an even function is an odd function, plus a constant, and the integral of an odd function is even.
Even and Odd Parts of Functions

\[ g(t) \]

\[ g_e(t) \]

\[ g_o(t) \]

\[ g(t) \]

\[ g_e(t) \]

\[ g_o(t) \]
Products of Even and Odd Functions

Two Even Functions

Even times Even is Even

\[ g_1(t) \times g_2(t) \]
Products of Even and Odd Functions

An Even Function and an Odd Function

Even times Odd is Odd

\[ g_1(t) \quad g_2(t) \]

\[ g_1(t)g_2(t) \]
Products of Even and Odd Functions

An Even Function and an Odd Function

Even times Odd is Odd

\[ g_1(t) \]

\[ g_2(t) \]

\[ g_1(t)g_2(t) \]
Products of Even and Odd Functions

Two Odd Functions

Odd times Odd is Even

\[ g_1(t)g_2(t) \]
Integrals of Even and Odd Functions

Even Function

\[ g(t) \]

Area #1 = Area #2

\[ \int_{-a}^{a} g(t) \, dt = 2 \int_{0}^{a} g(t) \, dt \]

Odd Function

\[ g(t) \]

Area #1 = - Area #2

\[ \int_{-a}^{a} g(t) \, dt = 0 \]

The integral of an odd function, over limits that are symmetrical about zero, is zero.
Integrals of Even and Odd Functions

Evaluate the integral

\[ I = \int_{-10}^{10} 4 \text{rect}(t/8)e^{j2\pi t/16} \, dt \]

\[ I = 4 \int_{-4}^{4} \left[ \cos(\pi t / 8) + j\sin(\pi t / 8) \right] \, dt = 8 \int_{0}^{4} \cos(\pi t / 8) \, dt + j4 \int_{4}^{4} \sin(\pi t / 8) \, dt \]

\[ I = 8 \left[ \frac{\sin(\pi t / 8)}{\pi / 8} \right]_{0}^{4} = \frac{64}{\pi} [1 - 0] = \frac{64}{\pi} \approx 20.372 \]
Periodic Signals

If a function $g(t)$ is periodic, $g(t) = g(t + nT)$ where $n$ is any integer and $T$ is a period of the function. The minimum positive value of $T$ for which $g(t) = g(t + T)$ is called the fundamental period $T_0$ of the function. The reciprocal of the fundamental period is the fundamental frequency $f_0 = 1 / T_0$.

A function that is not periodic is aperiodic.
Sums of Periodic Functions

The fundamental period of the sum of periodic functions is the least common multiple of the fundamental periods of the individual functions summed. If the least common multiple is infinite, the sum function is aperiodic.
Sums of Periodic Functions

Another way to find the fundamental period of a sum of periodic signals is through the fundamental frequency. The fundamental frequency of the sum of periodic functions is the greatest common divisor of the fundamental frequencies of the individual functions summed. If the greatest common divisor is zero, the sum function is aperiodic.

Find the fundamental period of the sum of

\[ 13\cos(200\pi t) \text{ and } 93\sin(240\pi t). \]

The fundamental frequency of \(13\cos(200\pi t)\) is 100 Hz.
The fundamental frequency of \(93\sin(240\pi t)\) is 120 Hz.
The greatest common divisor of 100 and 120 is 20.
Therefore the fundamental frequency of the sum is 20 Hz and the fundamental period of the sum is 1/20 second.
ADC Waveforms

Examples of waveforms which may appear in analog-to-digital converters. They can be described by a periodic repetition of a ramp returned to zero by a negative step or by a periodic repetition of a triangle-shaped function.
Transformer Voltage and Current

The voltage and current in a distribution transformer experiencing core saturation are shown above, voltage in blue and current in red. The current can be described by a sinusoid plus alternating periodic pulses which could be narrow triangles or even impulses as an approximate model.
Signal Energy and Power

The signal energy of a signal $x(t)$ is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$
Signal Energy and Power

\[ x(t) \]

\[ A \]

\[ B \]

\[ |x(t)|^2 \]

\[ A^2 \]

\[ B^2 \]

Area = Signal Energy
Signal Energy and Power

Find the signal energy of \( x(t) = \left[ 2 \text{rect}(t/2) - 4 \text{rect}\left(\frac{t+1}{4}\right) \right] u(t+2) \)

\[
E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} \left[ 2 \text{rect}(t/2) - 4 \text{rect}\left(\frac{t+1}{4}\right) \right]^2 u(t+2) \, dt
\]

\[
E_x = \int_{-2}^{\infty} \left[ 2 \text{rect}(t/2) - 4 \text{rect}\left(\frac{t+1}{4}\right) \right]^2 \, dt
\]

\[
E_x = \int_{-2}^{\infty} \left[ 4 \text{rect}^2\left(\frac{t+1}{4}\right) - 16 \text{rect}(t/2) \text{rect}\left(\frac{t+1}{4}\right) \right] \, dt
\]

\[
E_x = 4 \int_{-2}^{\infty} \text{rect}(t/2) \, dt + 16 \int_{-2}^{\infty} \text{rect}\left(\frac{t+1}{4}\right) \, dt - 16 \int_{-2}^{\infty} \text{rect}(t/2) \text{rect}\left(\frac{t+1}{4}\right) \, dt
\]

\[
E_x = 4 \int_{-1}^{1} dt + 16 \int_{-2}^{1} dt - 16 \int_{-1}^{1} dt = 8 + 48 - 32 = 24
\]
Signal Energy and Power

Some signals have infinite signal energy. In that case it is more convenient to deal with average signal power. The average signal power of a signal $x(t)$ is

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

For a periodic signal $x(t)$ the average signal power is

$$P_x = \frac{1}{T} \int_T |x(t)|^2 \, dt$$

where $T$ is any period of the signal.
Signal Energy and Power

A signal with finite signal energy is called an energy signal.

A signal with infinite signal energy and finite average signal power is called a power signal.
Signal Energy and Power

Find the average signal power of a signal \( x(t) \) with fundamental period 12, one period of which is described by

\[
x(t) = \text{ramp}(-t/5), \quad -4 < t < 8
\]

\[
P_x = \frac{1}{T} \int_{T} |x(t)|^2 \, dt = \frac{1}{12} \int_{-4}^{8} |\text{ramp}(-t/5)|^2 \, dt = \frac{1}{12} \int_{-4}^{0} \left(-\frac{t}{5}\right)^2 \, dt
\]

\[
P_x = \frac{1}{12} \int_{-4}^{0} \frac{t^2}{25} \, dt = \frac{1}{300} \left[ \frac{t^3}{3} \right]_{-4}^{0} = \frac{0 - (-64/3)}{300} = \frac{16}{225} \equiv 0.0711
\]
Magnitude and Phase of Complex Functions of Real Variables

\[ e^{-(3+j2.3)} = 0.0498 e^{-j2.3} \quad \text{or} \quad 0.0498 \angle -2.3, \quad e^{(2-j6)} = 7.3891 e^{-j6} = 7.3891 \angle 0.2832 \]

\[ \frac{100}{8 + j13} = 6.5512 e^{-j1.0191} = 6.5512 \angle -1.0191 \]

Fundamental Period of a Sum of Two Periodic Signals

\[ 3\sin(220\pi t) - 8\cos(120\pi t) \]

\[ \underbrace{f_0=110, \quad T_0=1/110}_{f_0=60, \quad T_0=1/60} \]

\[ f_0=\gcd(110,60)=10 \Rightarrow T_0=0.1 \]

Generalized Derivatives

\[ x(t) = \begin{cases} 4, & t < 3 \\ 7t, & t > 3 \end{cases} \Rightarrow x'(t) = 17\delta(t-3) + \begin{cases} 0, & t < 3 \\ 7, & t > 3 \end{cases} \]
Impulses and Periodic Impulses

Sampling Property \[ \int_{-8}^{22} 8 e^{4t} \delta(t-2) \, dt = 8 e^8 = 23,848 \quad \int_{11}^{82} 3 \sin(200t) \delta(t-7) \, dt = 0 \]

\[ \int_{-2}^{10} 39 t^2 \delta_4(t-1) \, dt = 39 \sum_{k=-\infty}^{\infty} \int_{-2}^{10} t^2 \delta(t-1-4k) \, dt = 39 \sum_{k=-\infty}^{\infty} \begin{cases} (4k+1)^2, & -2 < 4k+1 < 10 \\ 0, & \text{otherwise} \end{cases} \]

\[ = 39 \sum_{k=-\infty}^{\infty} \begin{cases} (4k+1)^2, & -3/4 < k < 9/4 \\ 0, & \text{otherwise} \end{cases} = 39(1+25+81) = 4173 \]

Equivalence Property \[ 7 \delta(t+4) \times (2t^2 + 5t + 1) = 7 \times \left[ 2(-4)^2 + 5(-4) + 1 \right] \delta(t+4) = 91 \delta(t+4) \]

Scaling Property \[ 5 \delta(3(t-1)) = (5/3) \delta(t-1) \quad -9 \delta_{11}(5t) = (-9/5) \delta_{11/5}(t) \]
Scaling and Shifting

\[ A = \frac{1}{2}, \ t_0 = -2, \ w = -0.5 \]

\[ A = -1.5, \ t_0 = 1, \ w = 2 \]
Even and odd functions

\[
x(t) = \frac{t^3 - 4t^2}{e^{jt^8}} \Rightarrow x_e(t) = \frac{x(t) + x(-t)}{2} = \frac{t^3 - 4t^2 + (-t)^3 - 4(-t)^2}{e^{jt^8} + e^{-jt^8}}
\]

\[
x_e(t) = \frac{t^3 e^{-jt^8} - 4t^2 e^{-jt^8} - t^3 e^{jt^8} - 4t^2 e^{jt^8}}{2} = \frac{t^3 (e^{-jt^8} - e^{jt^8}) - 4t^2 (e^{-jt^8} + e^{jt^8})}{2} = \frac{-j2t^3 \sin(8t) - 8t^2 \cos(8t)}{2} = -t^2 \left[ jt \sin(8t) + 4 \cos(8t) \right]
\]

\[
x_o(t) = \frac{t^3 - 4t^2}{e^{jt^8}} - \frac{(-t)^3 - 4(-t)^2}{e^{-jt^8}} = \frac{t^3 (e^{-jt^8} + e^{jt^8}) - 4t^2 (e^{-jt^8} - e^{jt^8})}{2} = t^2 \left[ t \cos(8t) + j4 \sin(8t) \right]
\]
Signal Energy and Signal Power

\[ x(t) = \begin{cases} |t| - 1, & |t| < 1 \\ 0, & \text{otherwise} \end{cases} \Rightarrow E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-1}^{1} (|t| - 1)^2 \, dt = 2 \int_{0}^{1} (t - 1)^2 \, dt \]

\[ E_x = 2 \int_{0}^{1} (t^2 - 2t + 1) \, dt = 2 \left[ \frac{t^3}{3} - t^2 + t \right]_0^1 = 2 \left( \frac{1}{3} - 1 + 1 \right) = \frac{2}{3} \]

\( x(t) \) is periodic and one period of \( x(t) \) is described by

\[ x(t) = t(1 - t), \quad 1 < t < 5 \]

\[ P_x = \frac{1}{T} \int_{T} |x(t)|^2 \, dt = \frac{1}{4} \int_{4}^{5} |t(1 - t)|^2 \, dt = \frac{1}{4} \int_{1}^{5} (t^2 - 2t^3 + t^4) \, dt \]

\[ P_x = \frac{1}{4} \left[ \frac{t^3}{3} - \frac{t^4}{2} + \frac{t^5}{5} \right]_1^5 
\[ P_x = \frac{1250 - 9375 + 18750 - 10 + 15 - 6}{4 \times 30} = \frac{1328}{15} = 88.5333 \]