Mathematical Description of Discrete-Time Signals
Sampling and Discrete Time

Sampling is the acquisition of the values of a continuous-time signal at discrete points in time. $x(t)$ is a continuous-time signal, $x[n]$ is a discrete-time signal.

$$x[n] = x(nT_s)$$

where $T_s$ is the time between samples.
Sampling and Discrete Time

\[ g(t) \]

\[ g[n] \]

\[ T_s \]
Sinusoids

Unlike a continuous-time sinusoid, a discrete-time sinusoid is not necessarily periodic. If it is periodic, its period must be an integer. If a sinusoid has the form \( g[n] = A \cos(2\pi F_0 n + \theta) \) then \( F_0 \) must be a ratio of integers (a rational number) for \( g[n] \) to be periodic. If \( F_0 \) is rational in the form \( q / N_0 \) \((q \text{ and } N_0 \text{ integers})\) in which all common factors in \( q \) and \( N_0 \) have already been cancelled, then the fundamental period of the sinusoid is \( N_0 \), not \( N_0 / q \) (unless \( q = 1 \)). Therefore, the general form of a periodic sinusoid with fundamental period \( N_0 \) is \( g[n] = A \cos(2\pi nq / N_0 + \theta) \).
Sinusoids

\[
x[n] = \sin(2\pi F_0 n), F_0 = \frac{1}{16} \quad \text{Periodic}
\]

\[
x[n] = \sin(2\pi F_0 n), F_0 = \frac{2}{16} \quad \text{Periodic}
\]

\[
x[n] = \sin(2\pi F_0 n), F_0 = \frac{11}{16} \quad \text{Periodic}
\]

\[
x[n] = \sin(2\pi F_0 n), F_0 = \frac{\pi}{16} \quad \text{Aperiodic}
\]
Sinusoids
An Aperiodic Sinusoid

\[ x[n] \]
Sinusoids
Two sinusoids whose analytical expressions look different,
\[ g_1[n] = A \cos(2\pi F_{01} n + \theta) \quad \text{and} \quad g_2[n] = A \cos(2\pi F_{02} n + \theta) \]
may actually be the same. If
\[ F_{02} = F_{01} + m, \quad \text{where} \ m \ \text{is an integer} \]
then (because \( n \) is discrete time and therefore an integer),
\[ A \cos(2\pi F_{02} n + \theta) = A \cos(2\pi (F_{01} + m) n + \theta) \]
\[
A \cos(2\pi F_{02} n + \theta) = A \cos \left( 2\pi F_{01} n + 2\pi mn + \theta \right) = A \cos(2\pi F_{01} n + \theta)
\]
(Example on next slide)
Sinusoids

\[ g_1[n] = \cos\left(\frac{2\pi n}{5}\right) \]

\[ g_2[n] = \cos\left(\frac{12\pi n}{5}\right) \]
Sinusoids

\[ x[n] = \cos(2\pi F n) \]  Dashed line is \( x(t) = \cos(2\pi F t) \)
Exponentials

The form of the exponential is

\[ x[n] = Az^n \quad \text{or} \quad x[n] = Ae^{\beta n} \quad \text{where} \quad z = e^\beta \]

Preferred

Real \( z \)

Complex \( z \)

\(|z| < 1\)

\(|z| > 1\)

\[ z > 1 \]

\[ z < -1 \]

\[ 0 < z < 1 \]

\[ -1 < z < 0 \]
The Unit Impulse Function

\[
\delta[n] = \begin{cases} 
1 & , \\ 0 & , n \neq 0
\end{cases}
\]

The discrete-time unit impulse (also known as the “Kronecker delta function”) is a function in the ordinary sense (in contrast with the continuous-time unit impulse). It has a sampling property,

\[
\sum_{n=-\infty}^{\infty} A\delta[n-n_0]x[n] = Ax[n_0]
\]

but no scaling property. That is,

\[
\delta[n] = \delta[an] \text{ for any non-zero, finite integer } a.
\]
The Unit Sequence Function

\[ u[n] = \begin{cases} 
1 &, n \geq 0 \\
0 &, n < 0 
\end{cases} \]
The Signum Function

\[ \text{sgn}[n] = \begin{cases} 
1 & , n > 0 \\
0 & , n = 0 = 2u[n] - \delta[n] - 1 \\
-1 & , n < 0 
\end{cases} \]
The Unit Ramp Function

\[ \text{ramp}[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases} = n u[n] = \sum_{m=-\infty}^{n} u[m-1] \]
The Periodic Impulse Function

\[ \delta_N[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN] \]
Scaling and Shifting Functions

Let \( g[n] \) be graphically defined by the graph below and

\[ g[n] = 0, \quad |n| > 15 \]
Scaling and Shifting Functions

Time shifting \( n \rightarrow n + n_0, \ n_0 \ \text{an integer} \)
Scaling and Shifting Functions

Time compression

\[ n \rightarrow Kn \]

\( K \) an integer > 1
Scaling and Shifting Functions

**Time expansion** \( n \rightarrow n / K, K > 1 \)

For all \( n \) such that \( n / K \) is an integer, \( g[n / K] \) is defined.

For all \( n \) such that \( n / K \) is not an integer, \( g[n / K] \) is **not defined**.
Scaling and Shifting Functions

There is a form of time expansion that is useful. Let

\[ y[n] = \begin{cases} 
  x[n/m], & n/m \text{ an integer} \\
  0, & \text{otherwise}
\end{cases} \]

All values of \( y \) are defined.

This type of time expansion is actually used in some digital signal processing operations.
Differencing

Backward Differences

Forward Differences
Accumulation

\[ g[n] = \sum_{m=-\infty}^{n} h[m] \]
Even and Odd Signals

\[ g[n] = g[-n] \quad \text{Even Function} \]
\[ g[n] = -g[-n] \quad \text{Odd Function} \]

Even Function
\[ g_e[n] = \frac{g[n] + g[-n]}{2} \]
Odd Function
\[ g_o[n] = \frac{g[n] - g[-n]}{2} \]
Products of Even and Odd Functions

Two Even Functions

\[ g_1[n] \times g_2[n] \]
Products of Even and Odd Functions

An Even Function and an Odd Function

\[ g_1[n] \]

\[ g_2[n] \]

\[ g_1[n] g_2[n] \]
Products of Even and Odd Functions

Two Odd Functions

\[ g_1[n] \]

\[ g_2[n] \]

\[ g_1[n] g_2[n] \]
Symmetric Finite Summation

Even Function

\[ g[n] \]

Sum #1 = Sum #2

\[ \sum_{n=-N}^{N} g[n] = g[0] + 2 \sum_{n=1}^{N} g[n] \]

Odd Function

\[ g[n] \]

Sum #1 = - Sum #2

\[ \sum_{n=-N}^{N} g[n] = 0 \]
Periodic Functions

A **periodic** function is one that is invariant to the change of variable $n \rightarrow n + mN$ where $N$ is a **period** of the function and $m$ is any integer.

The minimum positive integer value of $N$ for which $g[n] = g[n + N]$ is called the **fundamental period** $N_0$. 
Periodic Functions

Find the fundamental period of

\[ x[n] = \cos\left(\frac{\pi n}{18}\right) + \sin\left(\frac{10\pi n}{24}\right) \]

\[ x[n] = \cos\left(\frac{2\pi n}{36}\right) + \sin\left(\frac{2\pi n(5/24)}{N_0=24}\right) \]

\[ N_0 = \text{LCM}(36,24) = 72 \]

Find the fundamental period of

\[ x[n] = \cos\left(\frac{5\pi n}{13}\right) + \sin\left(\frac{8\pi n}{39}\right) \]

\[ x[n] = \cos\left(\frac{2\pi n(5/26)}{N_0=26}\right) + \sin\left(\frac{2\pi n(4/39)}{N_0=39}\right) \]

\[ N_0 = \text{LCM}(26,39) = 78 \]
Signal Energy and Power

The signal energy of a signal $x[n]$ is

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$
Signal Energy and Power

\[ x[n] = 0 \quad n > 31 \]

\[ \sum |x[n]|^2 \quad \text{Signal Energy} \]
Signal Energy and Power

Find the signal energy of

\[ x[n] = \begin{cases} (5/3)^{2n}, & 0 \leq n < 8 \\ 0, & \text{otherwise} \end{cases} \]

\[
E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{7} \left( \frac{5}{3} \right)^{2n} = \sum_{n=0}^{7} \left( \frac{5}{3} \right)^4^n
\]

Using

\[
\sum_{n=0}^{N-1} r^n = \begin{cases} N, & r = 1 \\ \frac{1 - r^N}{1 - r}, & r \neq 1 \end{cases}
\]

\[
E_x = \frac{1 - \left( \frac{5}{3} \right)^4}{1 - \left( \frac{5}{3} \right)^4} \approx 1.871 \times 10^6
\]
Signal Energy and Power

Some signals have infinite signal energy. In that case it is usually more convenient to deal with average signal power. The average signal power of a signal $x[n]$ is

$$P_x = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} |x[n]|^2$$

For a periodic signal $x[n]$ the average signal power is

$$P_x = \frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2$$

(The notation $\sum_{n=\langle N \rangle}$ means the sum over any set of consecutive $n$'s exactly $N$ in length.)
Signal Energy and Power

Find the average signal power of

\[ x[n] = 2 \text{sgn}[n] - 4 \]

\[ P_x = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} |x[n]|^2 = \lim_{x \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} |2 \text{sgn}[n] - 4|^2 \]

\[ P_x = \lim_{N \to \infty} \frac{1}{2N} \left\{ 4 \sum_{n=-N}^{N-1} \text{sgn}^2[n] + 16 \sum_{n=-N}^{N-1} 1 - 16 \sum_{n=-N}^{N-1} \text{sgn}[n] \right\} \]

\[ P_x = \lim_{N \to \infty} \frac{1}{2N} \left\{ 4(2N - 1) + 32N - 16(-1) \right\} = 20 \]
Signal Energy and Power

A signal with finite signal energy is called an energy signal.

A signal with infinite signal energy and finite average signal power is called a power signal.
Fundamental Period of a Sum of Two Periodic Signals
\[
\delta_{14}[n] - 6 \delta_8[n]
\]
\[
\begin{array}{c}
N_0=14 \\
N_0=8 \\
N_0=\text{LCM}(14,8)=56
\end{array}
\]
\[-2 \cos\left(\frac{3\pi n}{12}\right) + 11 \cos\left(\frac{14\pi n}{10}\right) = -2 \cos\left(\frac{2\pi n}{1/8}\right) + 11 \cos\left(\frac{2\pi n}{7/10}\right)
\]
\[
\begin{array}{c}
N_0=8 \\
N_0=10 \\
N_0=\text{LCM}(8,10)=40
\end{array}
\]

Impulses and Periodic Impulses
\[
\sum_{n=-18}^{33} 38n^2 \delta[n+6] = 1368, \quad \sum_{n=-4}^{7} -12(0.4)^n u[n] \delta_3[n] = -12\left[(0.4)^0 + (0.4)^3 + (0.4)^6\right] = -12.8172
\]

Equivalence Property  \(27(0.3)^n \delta[n-3] = 27(0.3)^3 \delta[n-3] = 0.729 \delta[n-3]\)

Scaling Property  \(13\delta[3n] = 13\delta[n]\), (No scaling property for discrete-time impulses)

\[
22 \delta_3[4n] = 22 \sum_{k=-\infty}^{\infty} \delta[4n-3k] = \begin{cases} 22 & 4n = 3k \\ 0 & \text{otherwise} \end{cases}
\]

Since \(k\) is an integer, impulses occur only where \(4n/3\) is an integer

\[
\begin{array}{cccccccc}
  n & 0 & 1 & 2 & 3 & 4 & \ldots \\
22 \delta_3[4n] & 22(k=0) & 0(4/3 \neq k) & 0(8/3 \neq k) & 22(k=1) & 0(16/3 \neq k) & \ldots
\end{array}
\]
Signal Energy and Signal Power

\[ x[n] = n(-1.3)^n (u[n] - u[n-4]) \Rightarrow E_x = \sum_{n=-\infty}^{\infty} |n(-1.3)^n (u[n] - u[n-4])|^2 \]

\[ E_x = \sum_{n=0}^{3} n^2 (1.3)^{2n} = 0 + 1.3^2 + 4 \times 1.3^4 + 9 \times 1.3^6 = 56.5557 \]

\[ x[n] \] is periodic and one period of \( x[n] \) is described by
\[ x[n] = n(1-n) \text{, } 3 \leq n < 6 \]

\[ P_x = \frac{1}{N} \sum_{n=\langle N \rangle}^{\langle N \rangle} |x[n]|^2 \]
\[ x[n] \]
\[ \begin{array}{cccc}
   & -6 & -12 & -20 \\
   \end{array} \]

\[ P_x = \frac{1}{3} [36 + 144 + 400] = \frac{580}{3} = 193.333\ldots \]