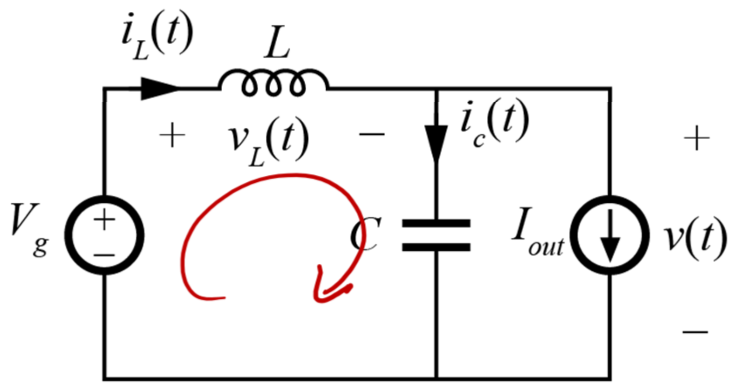
Formulation of State Space Matrices

- Network theoretic approaches exist for the formulation of state space equations
- Available commercial tools, e.g. PLECS expose solved state space descriptions to the user through command-line interface

$$\begin{bmatrix} B_{v_3} & B_{v_4} & B_{v_5} & B_{v_6} & B_{v_7} & B_{v_8} & B_{v_9} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Q_{i_1} & Q_{i_2} & Q_{i_3} & Q_{i_4} & Q_{i_5} & Q_{i_6} & Q_{i_7} \\ -U_3 & \cdot & G_{35} & G_{36} & \cdot & G_{38} & \cdot & \cdot & G_{32} & \cdot & G_{34} & G_{3i_5} & \cdot & \cdot \\ \cdot & -U_4 & G_{45} & G_{46} & \cdot & G_{48} & \cdot & \cdot & G_{42} & \cdot & G_{44} & G_{4i_5} & \cdot & \cdot \\ \cdot & \cdot & Y_C(D) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -Z_{LR}(D) & \cdot & \cdot \\ \cdot & \cdot & G_{65} & G_{66} & \cdot & G_{68} & \cdot & \cdot & G_{62} & \cdot & G_{64} & G_{6i_5} & -U_6 & \cdot \\ \cdot & \cdot & G_{75} & G_{76} & \cdot & G_{78} & \cdot & \cdot & G_{72} & \cdot & G_{74} & G_{7i_5} & \cdot & -U_7 \end{bmatrix} \begin{bmatrix} v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} B_{v_1} & B_{v_2} & \cdot & \cdot \\ \cdot & \cdot & Q_{i_8} & Q_{i_9} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ j_8 \\ j_9 \end{bmatrix}.$$

L. O. Chua and P. M. Lin, Chapter 8, *Computer-Aided Analysis of Electronic Circuits: Algorithms & Computational Techniques*, 1975
E. Purslow, "Solvability and analysis of linear active networks by use of the state equations," in *IEEE Transactions on Circuit Theory*, 1970

Example State Space Parsing



$$\begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_g & I_{out} \end{bmatrix}$$

simple case:

"well-formed" LTI circuit

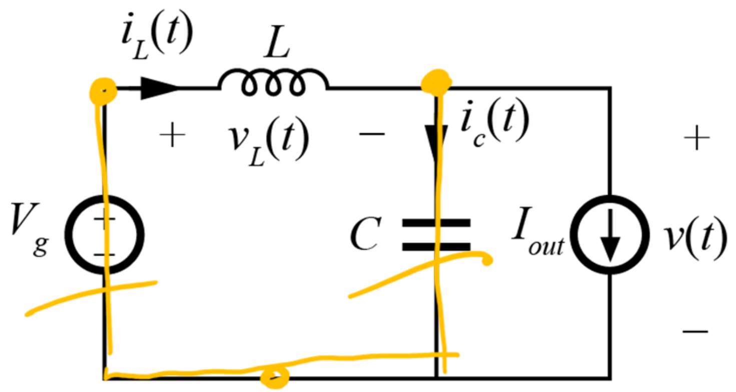
- All C & L are states
- No resisters in the circuit

Goal: express

$v_L = f(v_c, V_g) \rightarrow$ KVL w/ V_x, v_c & 1 other element

$i_c = f(i_L, I_{out}) \rightarrow$ KCL w/ I_x, i_L & 1 other element

Component Partitioning



KVL — KCL

→ V — C — R — L — I

Graph / Network Theory

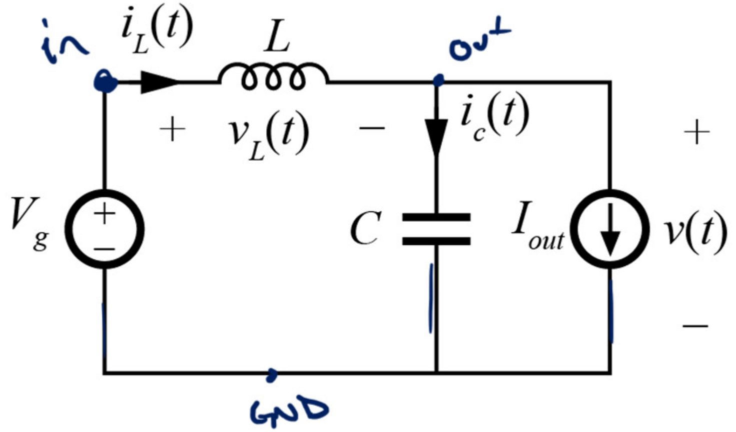
Tree: subset of all circuit elements ("branches") that includes every node but has no loops

co-Tree: All elements not in the tree ("links")

- Tree + any one link forms a loop

→ If we form a tree with only V & C components can write every voltage on link element as $\sum V_x$ & V_c

Incidence Matrix



Components

Nodes

$$\begin{array}{c|cccc} & V_g & C & L & I_{out} \\ \hline \text{in} & 1 & \phi & 1 & \phi \\ \text{GND} & -1 & -1 & \phi & -1 \\ \text{out} & \phi & 1 & -1 & 1 \end{array}$$

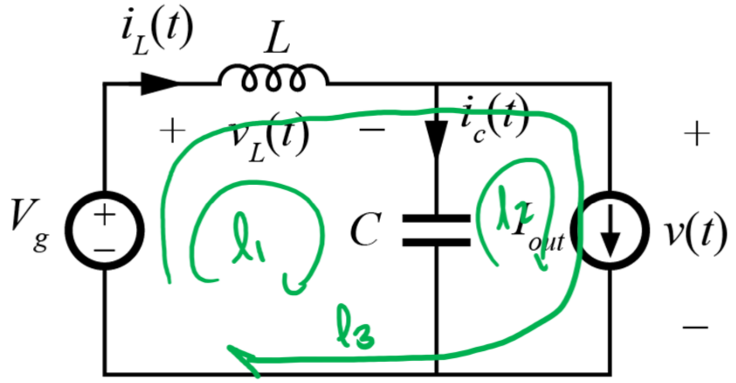
→ Incidence Matrix A_0

rows tell all of the components connected at each node, so

$$\boxed{A_0 i = \phi}, \qquad i = \begin{bmatrix} I_g \\ i_C \\ i_L \\ I_{out} \end{bmatrix}$$

All KCL at each node

Loop Matrix



$B_0 v = \emptyset$

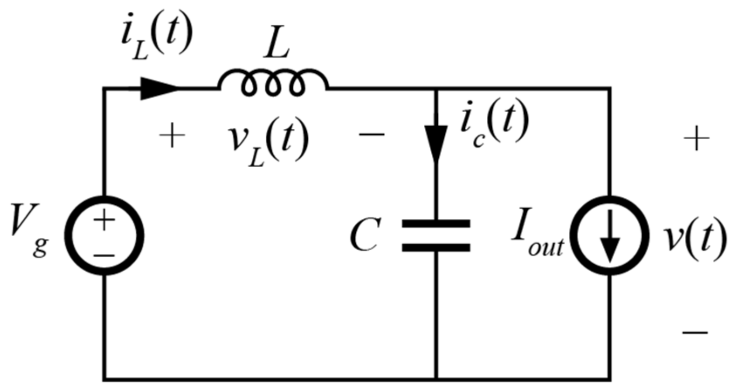
$v = \begin{bmatrix} v_g \\ v_c \\ v_L \\ v_{out} \end{bmatrix}$

components

loops

$$\begin{array}{c} \\ l_1 \\ l_2 \\ l_3 \end{array} \begin{array}{c} \begin{array}{cccc} v_g & c & L & I_{out} \end{array} \\ \begin{bmatrix} -1 & 1 & 1 & \emptyset \\ 0 & -1 & 0 & 1 \\ -1 & \emptyset & 1 & 1 \end{bmatrix} \end{array} = B_0$$

Selecting a Tree



$$A_0 = \begin{array}{c} \\ \text{in} \\ \text{GND} \\ \text{out} \end{array} \begin{array}{c} \begin{array}{cccc} V_g & C & L & I_{out} \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ \cancel{-1} & \cancel{-1} & \cancel{0} & \cancel{-1} \\ 0 & 1 & -1 & 1 \end{array}\right] \end{array}$$

Reduce A_0 to row echelon form

- In MATLAB rref()
- Apply elementary row operations to generate an upper triangular matrix with 1 for all pivots

Tree ←

$$A = \begin{array}{c} \begin{array}{cc|cc} \boxed{V_g \quad C} & & L & I_{out} \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \end{array}$$

components (columns) associated with first non-zero element in each row (pivots) are the tree

Partitioning

$$A = \begin{array}{cc} \text{tree} & \text{co-Tree} \\ \left[\, A_T \right. & \left. \vdots \; A_L \,\right] \end{array}$$

$$B = [\, B_T \;\vdots\; B_L \,]$$

In simplest case:

$A_T = I \rightarrow$ get an equation for current through all Tree components as a function of co-Tree currents

$B_L = I \rightarrow$ get equation for voltage on all co-Tree compents as function of only tree voltages

(flip columns before rref)

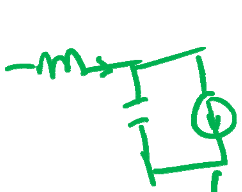
$(\text{Any row of } A) * (\text{Any row of } B)^T = \phi$, $AB^T = \phi$

$$\boxed{D = A_T^{-1} A}$$

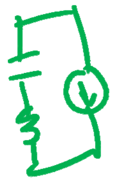
if loop & node don't share any elements

$=$ zero ✓

$=$ zero ✓



zero ✓



$=$ zero ✓

$AB^T = \phi$

$$[A_T \;\vdots\; A_L] \begin{bmatrix} B_T^T \\ \hline B_L^T \end{bmatrix} = \phi$$

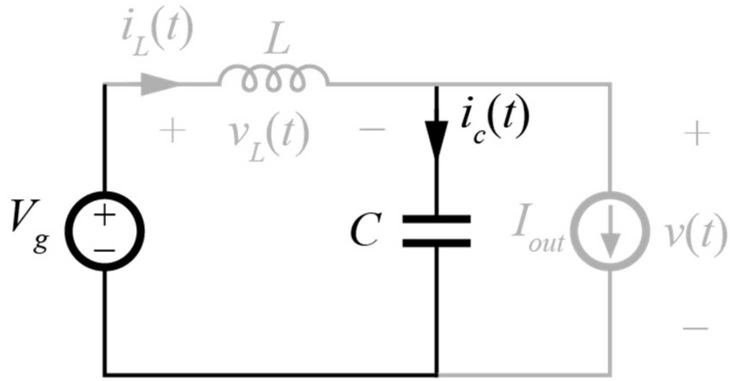
$$B = [\, B_T \;\vdots\; I \,]$$

$$= [\, (-A_T^{-1} A_L)^T \;\vdots\; I \,]$$

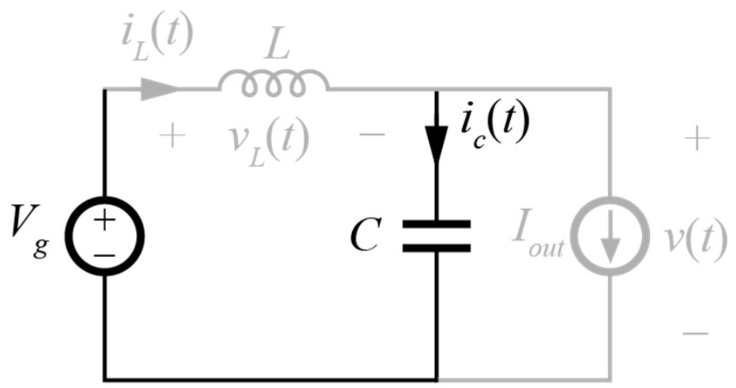
$$[A_T \;\vdots\; A_L] \begin{bmatrix} B_T^T \\ \hline I \end{bmatrix} = \phi = A_T B_T^T + A_L = \phi$$

$$\boxed{B_T^T = -A_T^{-1} A_L}$$

Loop and Cutset Matrices



Final State Space



$Bv = \emptyset$

$Di = \emptyset$

I

Components

$$\begin{array}{cc} & \begin{array}{cccc} V_g & C & L & I_{out} \end{array} \\ \boldsymbol{B} = & \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

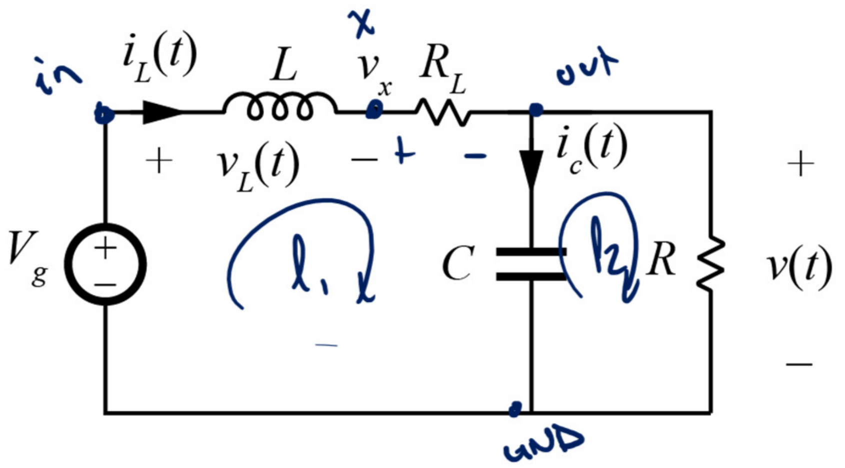
I

K B A

$v_L + v_c - v_g = \emptyset \rightarrow v_L = L\frac{di}{dt} = v_g - v_c$

$i_c - i_L + I_{out} = \emptyset \rightarrow i_c = C\frac{dv}{dt} = -I_{out} + i_L$

Including Resistances



$$\begin{array}{c} \\ \text{in} \\ \text{x} \\ \text{out} \\ \text{gnd} \end{array} \begin{array}{c} \begin{array}{ccccc} V_g & C & R & R_L & L \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & \emptyset & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 & \emptyset \\ -1 & -1 & -1 & \emptyset & \emptyset \end{array}\right] \end{array} = A_0$$

Components

Nodes

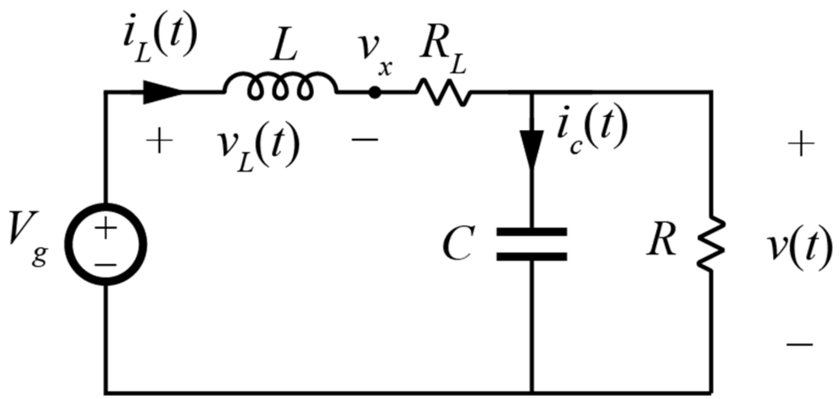
$\downarrow$ RREF

$$\begin{array}{c} \begin{array}{ccccc} & & R & R_L & \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \end{array} \begin{array}{l} \\ \leftarrow ② + ③ \\ \leftarrow ② \end{array}$$

$$A = \begin{array}{c} \begin{array}{ccccc} V_g & C & R_L & R & L \end{array} \\ \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right] \end{array} = \left[I \;\vdots\; A_L \right]$$

$$B = \begin{array}{c} \begin{array}{ccccc} V_g & C & R_L & R & L \end{array} \\ \left[\begin{array}{ccc|cc} 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 \end{array}\right] \end{array} = \left[B_T \;\vdots\; I \right]$$

Including Resistances



- Additional step to eliminate all resistor currents/voltages
  - solve voltage on tree resistors and current in link resistors

$$
\begin{array}{c}
\begin{array}{ccccc} V_g & C & RL & R & L \end{array} \\
\boldsymbol{D} = \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ \hline 0 & 0 & 1 & 0 & -1 \end{array}\right] = [\, \boldsymbol{I} \mid \boldsymbol{D}_L \,]
\end{array}
$$

$$
\boldsymbol{D}_L = \left[\begin{array}{c|c} \boldsymbol{D}_{EG} & \boldsymbol{D}_{EJ} \\ \hline \boldsymbol{D}_{RG} & \boldsymbol{D}_{RJ} \end{array}\right]
$$

$$
\begin{bmatrix} \boldsymbol{I}_{V,C} \\ \boldsymbol{V}_{I,L} \end{bmatrix} = \begin{bmatrix} -(\boldsymbol{D}_{EG}\boldsymbol{Z}^{-1}\boldsymbol{D}_{EG}^T) & \boldsymbol{D}_{EG}\boldsymbol{Z}^{-1}\boldsymbol{D}_{RG}^T\boldsymbol{Z}_R\boldsymbol{D}_{RJ} - \boldsymbol{D}_{EJ} \\ \boldsymbol{D}_{EJ}^T - \boldsymbol{D}_{RJ}^T\boldsymbol{Y}^{-1}\boldsymbol{D}_{RG}\boldsymbol{Y}_G\boldsymbol{D}_{EG}^T & -(\boldsymbol{D}_{RJ}^T\boldsymbol{Y}^{-1}\boldsymbol{D}_{RJ}) \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_{V,C} \\ \boldsymbol{I}_{I,L} \end{bmatrix}
$$

$$
\begin{bmatrix} i_g(t) \\ i_c(t) \\ v_L(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -\frac{1}{R} & 1 \\ 1 & -1 & -R_L \end{bmatrix} \begin{bmatrix} V_g \\ v_c(t) \\ i_L(t) \end{bmatrix}
$$