Integrator Frequency Response Comparison

CT: $G_I(s) = \frac{k}{s} \xrightarrow{s \to j\omega} G_I(s \to j\omega) = \frac{k}{j\omega} = \boxed{-j\frac{k}{\omega}}$

DT: $G_I(z) = \frac{kT_s}{2}\frac{1+z^{-1}}{1-z^{-1}} \xrightarrow{z \to e^{j\omega T_s}} G_I(z \to e^{j\omega T_s}) = \frac{kT_s}{2}\frac{1+e^{-j\omega T_s}}{1-e^{-j\omega T_s}} \frac{\left(\frac{1}{2j}e^{j\omega\frac{T_s}{2}}\right)}{\left(\frac{1}{2j}e^{j\omega T_s/2}\right)}$

$$= \frac{kT_s}{2}\frac{\frac{1}{2j}\left(e^{j\omega\frac{T_s}{2}} + e^{-j\omega\frac{T_s}{2}}\right)}{\frac{1}{2j}\left(e^{j\omega\frac{T_s}{2}} - e^{-j\omega\frac{T_s}{2}}\right)} = \frac{kT_s}{2}\frac{\frac{1}{j}\cos\left(\omega\frac{T_s}{2}\right)}{\sin\left(\omega\frac{T_s}{2}\right)} = \boxed{-j\frac{kT_s}{2}\frac{1}{\tan\left(\omega\frac{T_s}{2}\right)}}$$

$\tan(\theta) \approx \theta$ for $|\theta| \ll 1$

for $\left|\frac{\omega T_s}{2}\right| \ll 1$

$$G_I(z \to e^{j\omega T_s}) \approx -j\frac{k\cancel{T_s}}{\cancel{2}}\frac{1}{\omega\frac{\cancel{T_s}}{\cancel{2}}} = -j\frac{k}{\omega}$$

$$\boxed{\frac{1}{s} \longleftrightarrow \frac{T_s}{2}\frac{1+z^{-1}}{1-z^{-1}}} \qquad \boxed{s \longleftrightarrow \frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}}$$

conversion

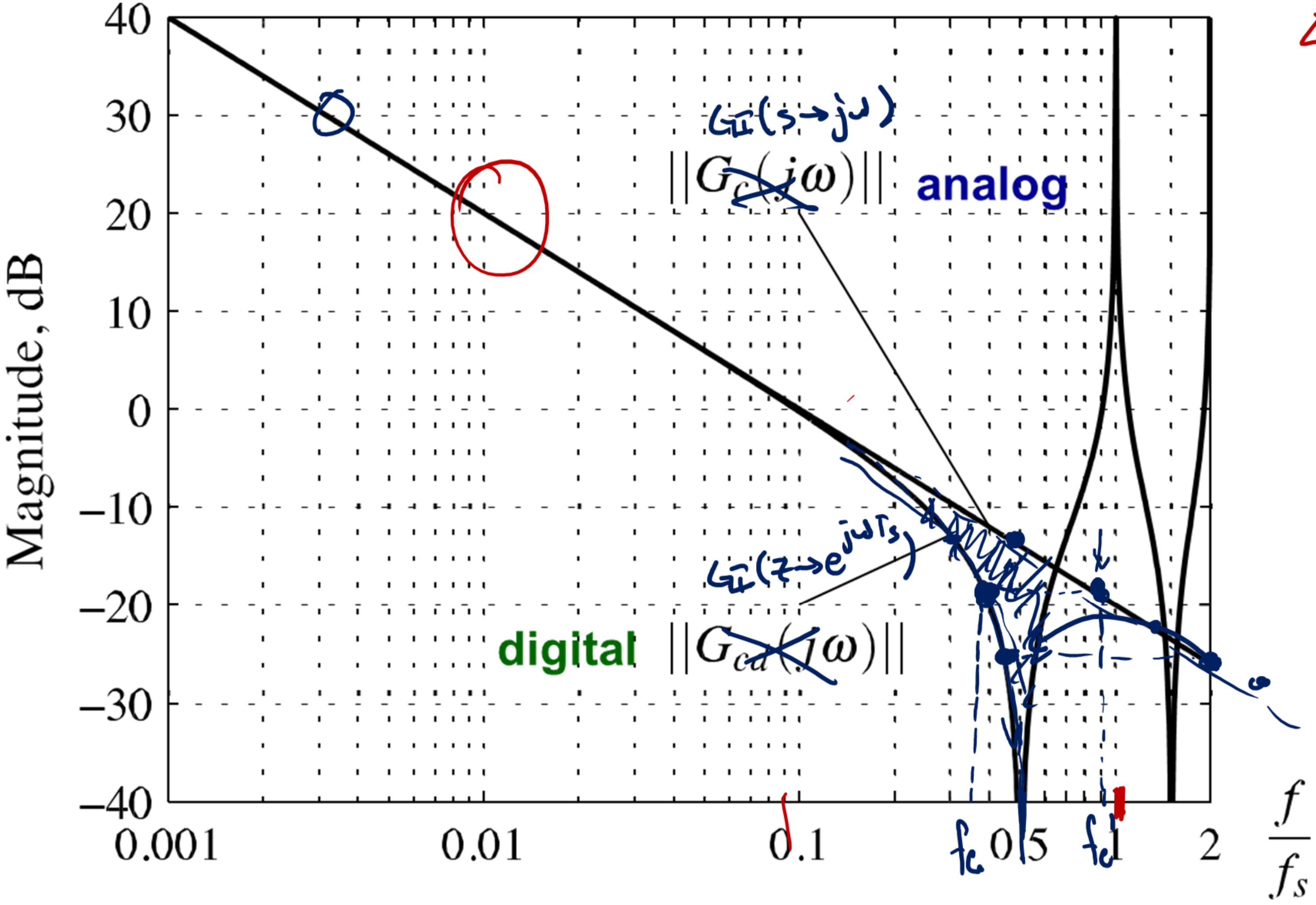
$\frac{k}{\omega} = \frac{kT_s}{2}\frac{1}{\tan\left(\omega\frac{T_s}{2}\right)} \rightarrow$ where is $\|G_I(s)\| = \|G_I(z)\|$

$$\omega_{CT} = \frac{2}{T_s}\tan\left(\omega_{DT}\frac{T_s}{2}\right)$$

Magnitude comparison

$\angle G_I(s) = -90^\circ$

$\angle G_I(z) = -90^\circ$



Example: $f_o = f_s/10$

DT Natural Response

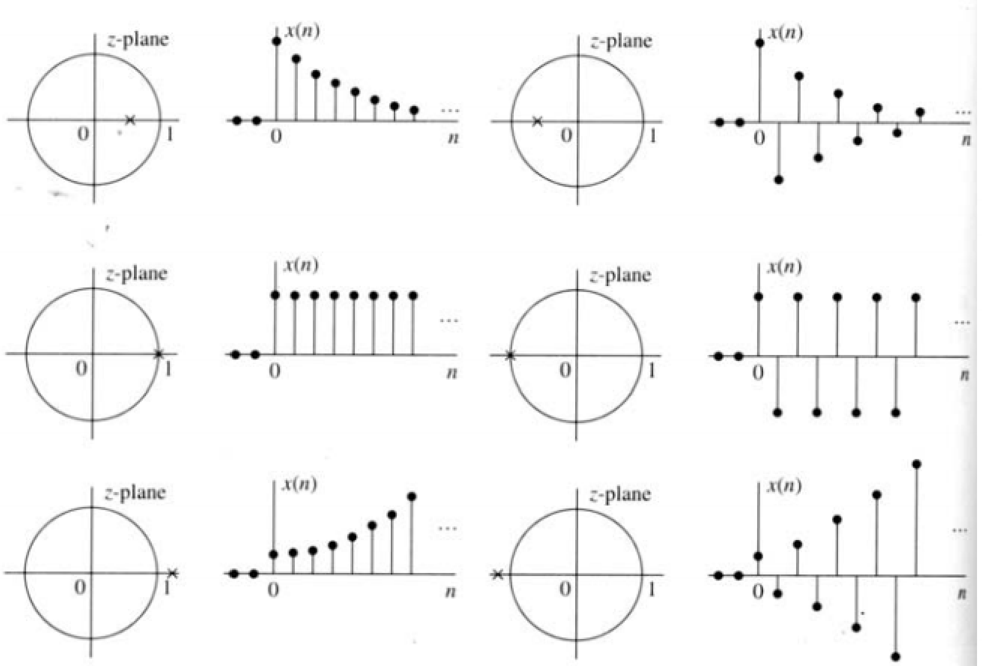


Figure 3.11 Time-domain behavior of a single-real pole causal signal as a function of the location of the pole with respect to the unit circle.

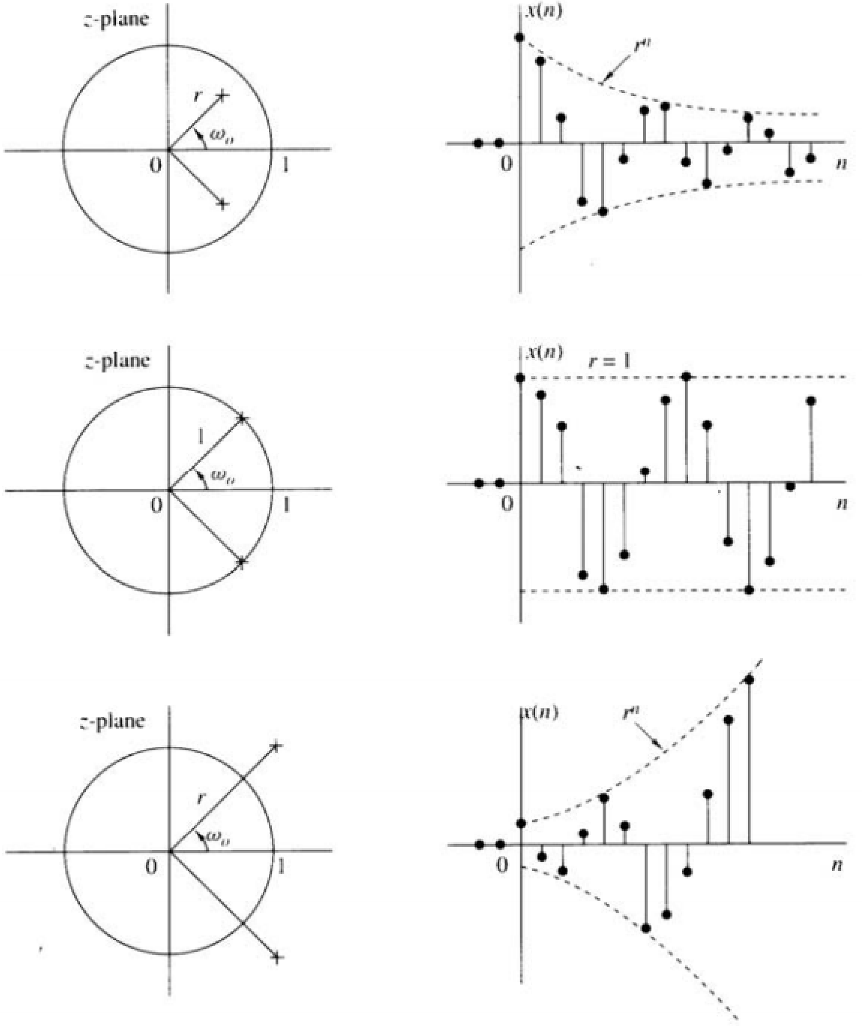


Figure 3.13 A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

Frequency Response

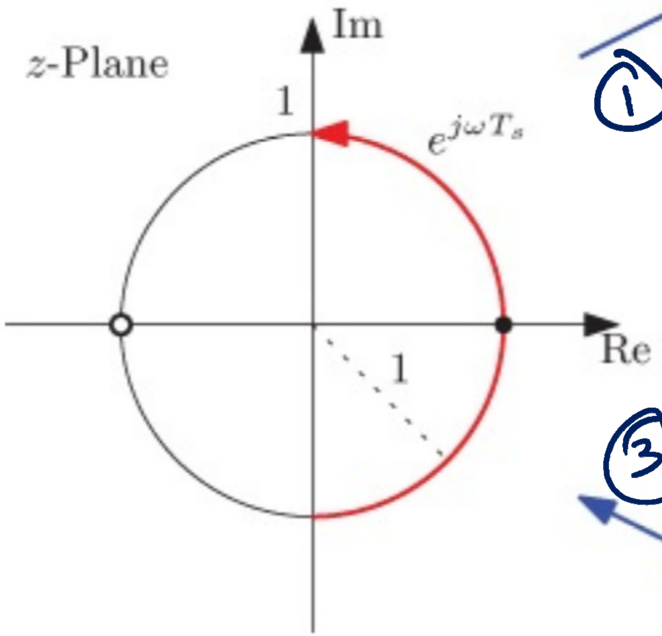
Mapping Between Domains

Laplace-like Domain

$\left\{ \begin{array}{l} G_{vd}(z) \\ G_{vg}(z) \end{array} \right.$

specific design criteria

2 $\omega' = \frac{2}{T_s}\tan\left(\omega\frac{T_s}{2}\right).$

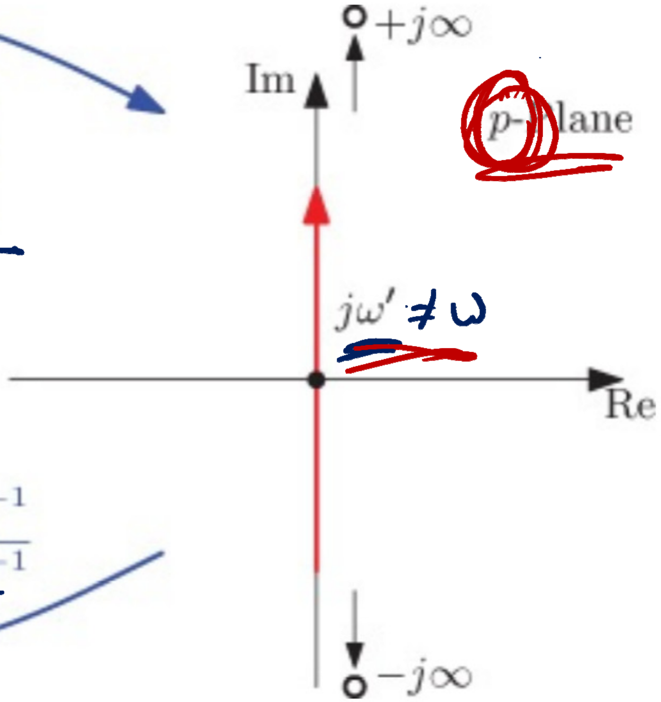


1 $z(p) = \dfrac{1 + p\frac{T_s}{2}}{1 - p\frac{T_s}{2}}$

DT → CT

3 $p(z) = \dfrac{2}{T_s}\dfrac{1 - z^{-1}}{1 + z^{-1}}$

DT ← CT



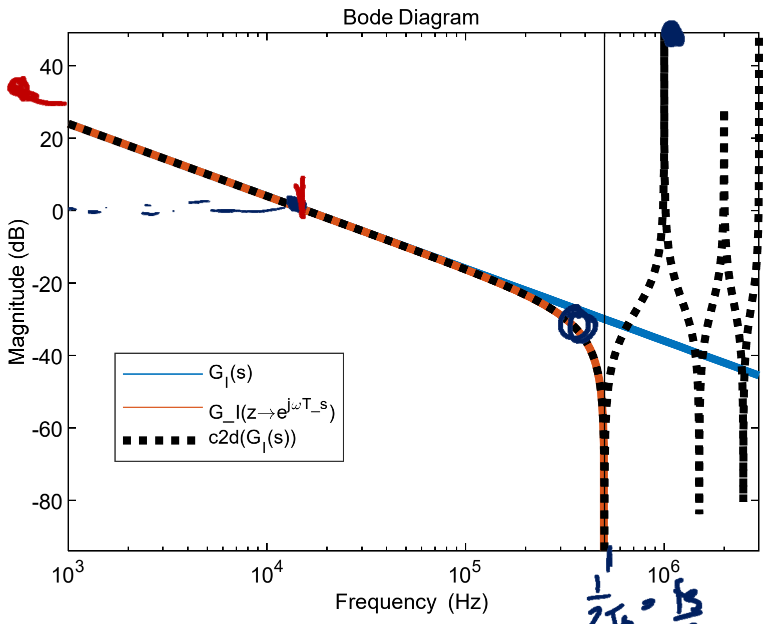
Matlab Examples

$T_s = 1\,\mu s$

$k = 10{,}000\,(2\pi)$

$\omega = 0$

```
bodemag(K/s, opts, {wmin, wmax});
hold on;
bodemag(c2d(K/s,Ts,'Tustin'), opts, {wmin, wmax});
plot(w/2/pi, 20*log10(abs(K*Ts/2*(1+exp(1j*w*Ts))./(1-exp(1j*w*Ts)))),':k');
```

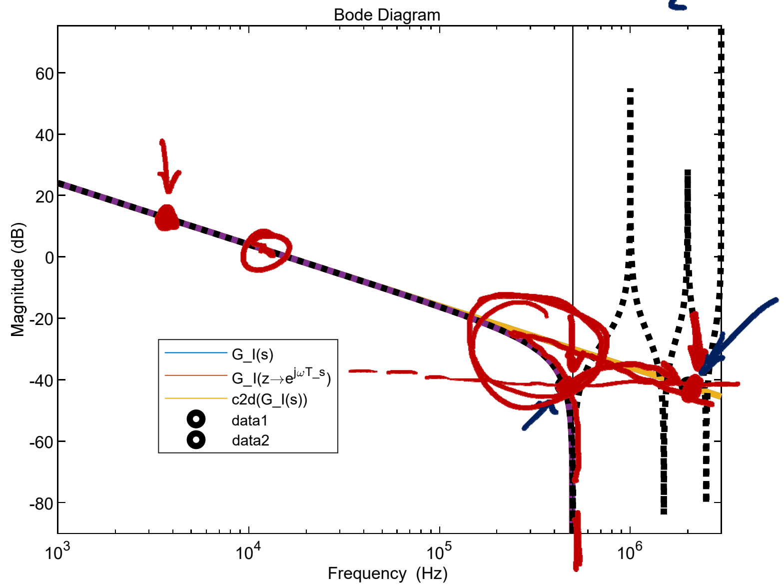


$\frac{1}{2T_s} = \frac{f_s}{2}$

```
wDT = 2*pi*450e3;
M = bode(K*Ts/2*(1+z^-1)/(1-z^-1), wDT)
plot(wDT/2/pi,20*log10(M), 'ok','LineWidth',3)
```

```
wCT = 2/Ts*tan(wDT*Ts/2);
```

```
M = bode(d2c(K*Ts/2*(1+z^-1)/(1-z^-1),'Tustin'), wCT)
plot(wCT/2/pi,20*log10(M), 'ok','LineWidth',3)
```



Prewarping

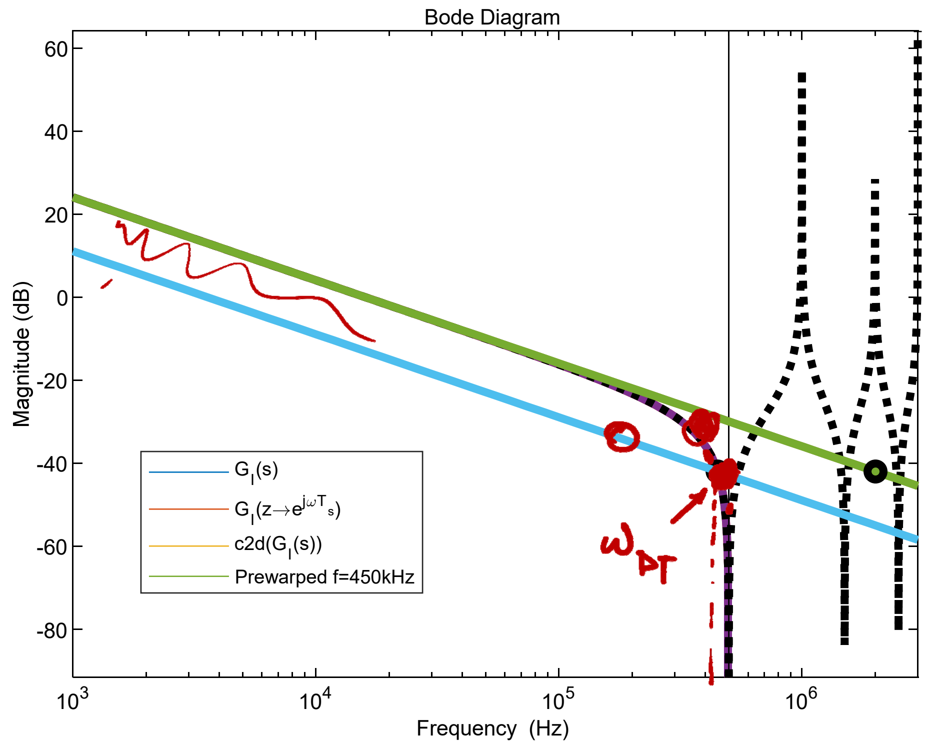
$\omega_{DT} = 2\pi 450\text{kHz}$

```
dopts = d2cOptions(PrewarpFrequency=wDT, method="Tustin");
bodemag(d2c(K*Ts/2*(1+z^-1)/(1-z^-1),dopts), opts, {wmin, wmax});
```

Prewarped conversion

$$p(z) = \frac{\omega^*}{\tan\left(\frac{\omega^* T_s}{2}\right)}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

ω^* = matching frequency



Compensator Design Approach

- Find $T_u(z)$ using Discrete time analysis (uncompensated loop gain)
- Map $T_u(z) \rightarrow T_u'(p)$ using 1
- Map any frequency specifications $\omega \rightarrow \omega'$ using 2
- Design $G_c'(p)$ to shape $T_u'(p)$ & specifications @ ω'
  - Using all same approaches as $G_c(s)$
- Map $G_c'(p)$ back to $G_c(z)$ using 3