$(sI-A)^{-1} \rightarrow$ Resolvent of $A$

$\Phi(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\} \rightarrow$ state transition matrix / Fundamental matrix

$$\underline{x}(t) = \underline{\underline{\Phi}}(t)\,\underline{x}_0 + \int_0^t \underline{\underline{\Phi}}(\tau)\,\underline{\underline{B}}\,\underline{u}(t-\tau)\,d\tau$$

$$= \underline{\underline{\Phi}}(t)\,\underline{x}_0 + \int_0^t \underline{\underline{\Phi}}(t-\tau)\,\underline{\underline{B}}\,\underline{u}(\tau)\,d\tau$$

solution to LTI equivalent w/in one subinterval

if all ICs are zero / ignored

$$\underline{x}(s) = (s\underline{\underline{I}}-\underline{\underline{A}})^{-1}\underline{\underline{B}}\,\underline{u}$$

$$\underline{Y}(s) = \underline{\underline{C}}\,\underline{X}(s) + \underline{\underline{D}}\,\underline{U}(s)$$

$$\underline{Y}(s) = C(sI-A)^{-1}Bu(s) + Du(s)$$

$$G(s) = \frac{y_i}{u_j}(s) = \boxed{C(sI-A)^{-1}B + D}$$

# Example Calculation

$$\boldsymbol{A}_I = \begin{bmatrix} -\frac{\overbrace{(r_{on}+R_L)}^{R_e = r_{on}+R_L}}{L} & 0 \\ 0 & 0 \end{bmatrix}$$

Let's calculate $\Phi(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\frac{R_e}{L} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s+\frac{R_e}{L} & 0 \\ 0 & s \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{\mathrm{Adj}(sI-A)}{\det(sI-A)} = \frac{1}{|sI-A|}\begin{bmatrix} s & 0 \\ 0 & s+\frac{R_e}{L} \end{bmatrix}$$

$$\det(sI-A) = |sI-A| = s\left(s+\frac{R_e}{L}\right) - 0$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s+R_e/L} & 0 \\ 0 & \frac{1}{s} \end{bmatrix}$$

$$\mathcal{L}^{-1}\{(sI-A)^{-1}\} = \begin{bmatrix} e^{-\frac{R_e}{L}t} & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for } t < 0$$

$$= \Phi(t)$$

if $\underline{u} = \underline{0}$

$$\underline{x}(t) = \underline{\Phi}(t)\, x_0$$

$$x(t) = \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix}$$

# State Transition Matrix

Find $\Phi(t)$ for a general case by looking at homogeneous (zero-input) response of the system.

$$\rightarrow \dot{x} = Ax \qquad\qquad \Phi = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

$$x = \Phi x_0$$

Assume that $x(t)$ can be represented by a power series

$$x(t) = (x_0 + x_1 t + x_2 t^2 + \cdots) = \sum_{i=0}^{\infty} x_i t^i$$

Then $\Phi(t)$ must also be representable as a power series

$$\Phi(t) = (P_0 + P_1 t + P_2 t^2 + \cdots) = \sum_{i=0}^{\infty} P_i t^i$$

we know immediately, $\boxed{P_0 = I}$

$$\dot{x} = Ax = \frac{d}{dt}\Phi x_0$$

$$\frac{d}{dt}\Phi(t)\, x_0 = [\cancel{0} + P_1 + 2P_2 t + 3P_3 t^2 + \cdots]\, x_0$$

$$\dot{x} = A\,x(t) = A\,\Phi(t)\,x_0 = \frac{d}{dt}\Phi\, x_0$$

$$A P_0 x_0 + A P_1 t\, x_0 + A P_2 t^2 x_0 + \vdots \qquad P_1 x_0 + 2P_2 t\, x_0 + 3P_3 t^2 x_0 + \vdots$$

$$P_1 x_0 \longrightarrow P_1 = A$$

$$2P_2 t\, x_0 \longrightarrow P_2 = \frac{1}{2}A^2$$

$$3P_3 t^2 x_0 \longrightarrow P_3 = \frac{1}{2}\cdot\frac{1}{3}A^3 = \frac{1}{3!}A^3$$

$$\vdots \longrightarrow P_4 = \frac{1}{4!}A^4$$

$$\vdots$$

$$P_n = \frac{1}{n!}A^n$$

$$\Phi(t) = P_0 + P_1 t + P_2 t^2 + \cdots$$

$$= I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$\Phi(t) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i \equiv e^{At} \rightarrow \text{Matrix exponential}$$

$$e^{at} = \sum_{i=0}^{\infty} \frac{1}{i!} (at)^i$$

$a \rightarrow$ scalar

so, $\boxed{\Phi(t) = e^{At}}$

# Properties of the Matrix Exponential

**Derivative**

$$\frac{d}{dt}e^{At} = Ae^{At}$$

**Nonvanishing Determinant**

$$\det e^{At} \neq 0$$

**Same-Matrix Product**

$$e^{At}e^{As} = e^{A(t+s)}$$

t & s, scalars

**Inverse**

$$\left(e^{At}\right)^{-1} = e^{-At}$$

**Commutative Product (1)**

$$AB = BA \implies e^{At}B = Be^{At}$$

Note that a matrix always commutes w/ itself A & A

**Commutative Product (2)**

$$AB = BA \implies e^{At}e^{Bt} = e^{(A+B)t}$$

if A & B commute

**Series Expansion**

$$e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!}A^n$$

**Decomposition**

$$e^{PBP^{-1}} = Pe^{B}P^{-1}$$

# Matrix Exponential

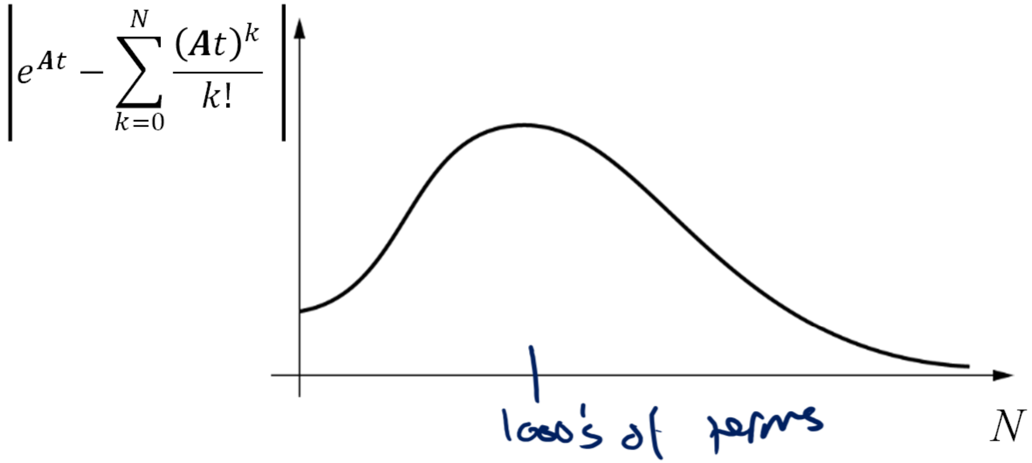
Matrix exponential defined by Taylor series expansion

$$e^{\boldsymbol{A}t} = I + \boldsymbol{A}t + \frac{(\boldsymbol{A}t)^2}{2!} + \cdots + \frac{(\boldsymbol{A}t)^N}{N!} = \sum_{k=0}^{N} \frac{(\boldsymbol{A}t)^k}{k!}$$

Summation always converges and is invertible,

$$e^{\boldsymbol{A}} e^{-\boldsymbol{A}} = \boldsymbol{I}$$

Well-known issue with convergence in many cases




**In MATLAB, expm(·) calculates matrix exponential**

– Using exp(·) will compute the element-by-element exponential

C. Moler and C. V. Loan, "Nineteen dubious ways to compute the exponential of a matrix," SIAM Review, vol. 20, pp. 801–836, 1978.

# Time-Domain Derivation

$$\dot{x} = Ax + Bu$$

multiply both sides by $e^{-At}$

$$e^{-At}\dot{x} = e^{-At}Ax + e^{-At}Bu$$

$$e^{-At}\dot{x} - e^{-At}Ax = e^{-At}Bu$$

$$e^{-At}\frac{d}{dt}x + \frac{d}{dt}\left(e^{-At}\right)x = e^{-At}Bu$$

$$\int_0^t \frac{d}{dt}\left(e^{-At}x\right) = \int_0^t e^{-At}Bu$$

$$e^{-At}x - e^{-A\emptyset}x_0 = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}$$

# Solution With DC Inputs

if $u(t) \approx U \rightarrow$ constant, for SMPS inputs are approximately constant w/in one switching subinterval

$$x(t) = e^{At}x_0 + \int_0^t e^{A\tau} B \underbrace{u(t-\tau)}_{U} d\tau$$

$$\boxed{x(t) = e^{At}x_0 + A^{-1}\left(e^{At} - I\right)BU}$$

$A^{-1}$ is needed to calculate the above expression

$$A^{-1}\left(e^{At} - I\right)BU$$

$$A^{-1}\left(\cancel{I} + At + \tfrac{1}{2}A^2t^2 + \dots - \cancel{I}\right)BU$$

$$\left(t + \tfrac{1}{2}At^2 + \tfrac{1}{6}A^2t^3 + \dots\right)BU \rightarrow \text{Bad way to calculate}$$

Can still be solved even if $A_i$ is singular