Augmented State Space Approach

$$x(t) = \boxed{e^{At}} x_0 + \boxed{\int_0^t e^{A(t-\tau)} B}\, u(\tau)\, d\tau \Longrightarrow x(t) = e^{At}x_0 + \cancel{A^{-1}(e^{At} - I)Bu}$$

u_i

p_i

Augmented State Matrix

$$\tilde{A} = \begin{array}{c|cc} & n_s & p_i \\ n_s & A & B \\ p_i & \emptyset & \emptyset \end{array} \longrightarrow e^{\tilde{A}t} = \begin{array}{c|cc} & n_s & p_i \\ n_s & \boxed{e^{At}} & \boxed{\int_0^t e^{A(t-\tau)} B\, d\tau} \\ p_i & \emptyset & I \end{array}$$

$$x(t) = \boxed{e^{At}} x_0 + \boxed{\left[\int_0^t e^{A(t-\tau)} B\, d\tau\right]} u$$

one option to apply:
just use $e^{\tilde{A}t}$ & extract matrices needed

second option for how to apply: solve for $\tilde{X}_{ss}$

in each subinterval, our augmented system is $\dot{\tilde{x}}(t) = \tilde{A}_i \tilde{x}$, $\tilde{x} = \begin{bmatrix} x \\ u \end{bmatrix}$

solution is $\tilde{x}(t) = e^{\tilde{A}_i t} \tilde{x}_{0,i}$ → homogeneous system $\dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}$

for k subintervals, $\tilde{x}(T_s) = \left(\prod_{i=k}^{1} e^{\tilde{A}_i t_i} \right) \tilde{x}_0$

$$\tilde{x}(T_s) = \left(\prod_{i=n}^{1} e^{A_i t_i}\right) \tilde{x}_0$$

in steady-state, $\tilde{x}(T_s) = \tilde{x}_0 = \tilde{x}_{ss}$

$$\boxed{\tilde{x}_{ss}\left(I - \prod_{i=n}^{1} e^{A_i t_i}\right) = \phi}$$

$\rightarrow$ find null space of $\left(I - \prod_{i=n}^{1} e^{A_i t_i}\right)$

in matlab $\underset{[k_1 \dots k_n]}{}= \text{null}(\Phi) \rightarrow$ gives a basis for the null space of Φ

$$\tilde{x}_{ss} = \begin{bmatrix} x_{ss} \\ u \end{bmatrix} \rightarrow \text{find } \alpha_1 k_1 + \alpha_2 k_2 + \dots \alpha_n k_n = \begin{bmatrix} x_{ss} \\ u \end{bmatrix} \rightarrow$$

Augmented State Space: Alternative Form

Alternative

$$\hat{A} = \begin{array}{c} \\ n_s \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} n_s & 1 \end{array} \\ \begin{bmatrix} A & Bu \\ \phi & \phi \end{bmatrix} \end{array} \rightarrow \quad \hat{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \rightarrow \quad \dot{\hat{x}} = \hat{A}\hat{x}$$

$$\hat{x}(t) = e^{\hat{A}t}\hat{x}$$

$$\downarrow$$

$$\begin{array}{c} \\ n_s \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} n_s & 1 \end{array} \\ \begin{bmatrix} e^{At} & \int_0^t e^{A(t-\tau)} B u \, dt \\ \phi & 1 \end{bmatrix} \end{array}$$

will only have one nullspace basis even for mutiple independent inputs

Computing Integrals

Very often, want $\frac{1}{T_s}\int_0^{T_s} x(\tau)\,d\tau$ or $\frac{1}{T_s}\int_0^{T_s} y(\tau)\,d\tau$ for things like P_{in}, P_{out}, η

if we want $\int_0^t x(\tau)d\tau = \int_0^t \left[e^{A\tau}x_0 + \int_0^\tau e^{A(\tau-\mu)}Bu\,d\mu\right]d\tau$

$$= A^{-1}\left(e^{At}-I\right)x_0 + \int_0^t A^{-1}\left(e^{At}-I\right)Bu\,d\tau$$

$$= A^{-1}\left(e^{At}-I\right)x_0 + A^{-1}\left(A^{-1}\left(e^{At}-I\right)-I\right)Bu$$

$\dot{x} = Ax + Bu$

$\hat{x} = \begin{bmatrix} x \\ x_{new} \end{bmatrix}$

$\dot{\hat{x}} = \hat{A}\begin{bmatrix} x \\ x_{new} \end{bmatrix} + \hat{B}u = \begin{bmatrix} \dot{x} \\ \dot{x}_{new} \end{bmatrix}$ will get integrated over time in system solution

$\hat{x}(t) = \int_0^t \dot{\hat{x}}\,dt$

so if $\dot{x}_{new} = x$ ← original states

$\hat{x}(t) = \begin{bmatrix} x \\ \int_0^t x\,d\tau \end{bmatrix}$

$$\hat{A} = \begin{array}{cc} & \begin{array}{cc} n_s & n_s \end{array} \\ \begin{array}{c} n_s \\ n_s \end{array} & \begin{bmatrix} A & \emptyset \\ I & \emptyset \end{bmatrix} \end{array}$$

$$\dot{\hat{x}} = \begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \hat{A}\hat{x} + \hat{B}\hat{u}$$

$$\hat{B} = \begin{bmatrix} B \\ \emptyset \end{bmatrix}$$

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this integral-computing augmentation cannot be used inside the steady-state solution because $\frac{1}{T_s}\int_0^{T_s} x(\tau)\,d\tau$ is generally <u>not</u> periodic

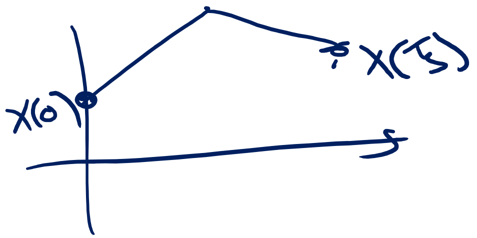
$$\hat{x}_0 = \begin{bmatrix} x_{ss} \\ \emptyset \end{bmatrix}$$

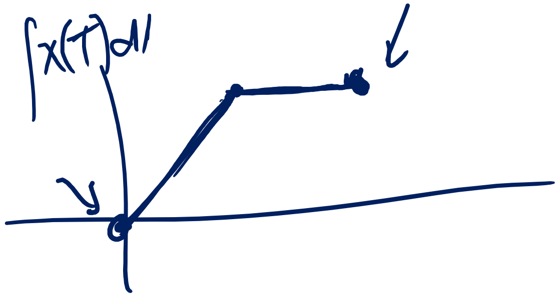
$\int_\emptyset^\emptyset x(\tau)\,d\tau]$

$$\hat{x}(T_s) = \begin{bmatrix} x_{ss} \\ \int_0^{T_s} x(\tau)\,d\tau \end{bmatrix}$$

if Augments used

$$\hat{x}(T_s) = \left(\prod_{i=n}^{1} e^{\hat{A} t_i}\right)\hat{x}_0$$





E.g. using Augment #1

$$\tilde{A}_i = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \rightarrow \tilde{x}_{ss}\left(I - \prod_{i=h}^{1} e^{\tilde{A}_i t_i}\right) = \phi \qquad \hat{x}_{ss} = \begin{bmatrix} x_{ss} \\ u \end{bmatrix}$$

$$\hat{A}_i = \begin{array}{c|c} & \begin{array}{ccc} n_s & p_i & n_s \end{array} \\ \begin{array}{c} n_s \\ p_i \\ n_s \end{array} & \begin{bmatrix} A & B & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} \end{array}$$

$$\hat{x}(T_s) = \left(\prod_{i=h}^{1} e^{\hat{A}_i t_i}\right) \hat{x}_0$$

$$\hat{x}_0 = \begin{bmatrix} x_{ss} \\ u \\ \phi \end{bmatrix}$$

$$\hat{x}(T_s) = \begin{bmatrix} x_{ss} \\ u \\ \boxed{\int_0^{T_s} x(\tau)\,d\tau} \end{bmatrix}$$