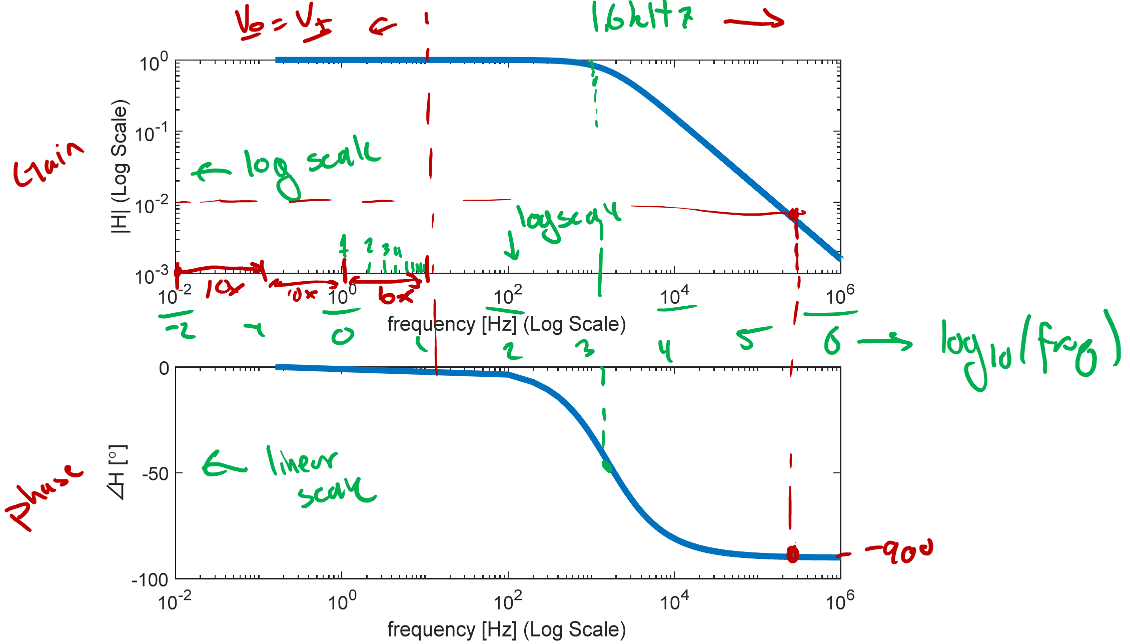
Bode Plot – Frequency Response

low-pass filter
(LPF)

R = 10Ω
C = 10nF

$\frac{1}{RC}$ = 1.6kHz



Fourier Series

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = \underline{a_0} + \sum_{k=1}^{\infty} \underline{a_k}\cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$f(t)$ can be expressed this way if

1. $f(t)$ is single-valued
2. $\int_{t_0}^{t_0+T_0} |f(t)|\, dt$ exists
3. $f(t)$ had finite discontinuities and max/min per period

Need to find a_0, a_k, b_k for some function $f(t)$

for a_0: $\boxed{a_0 = \frac{1}{T}\int_0^{T_0} f(t)\, dt}$ a_0 is average / DC value of $f(t)$

For a_k:

$n \in \mathbb{Z}^+$ $a_n \rightarrow$ look at $\frac{1}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\, dt$

plugging in Fourier series for $f(t)$:

$$\frac{1}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\, dt = \frac{1}{T_0}\int_0^{T_0}\left[a_0 + \sum_{k=1}^{\infty} a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)\right]\cos(n\omega_0 t)\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \cancel{a_0\cos(n\omega_0 t)}\, dt \xrightarrow{} 0 + \frac{1}{T_0}\int_0^{T_0}\sum_{k=1}^{\infty}\left[a_k\cos(k\omega_0 t)\cos(n\omega_0 t) + b_k\sin(k\omega_0 t)\cos(n\omega_0 t)\right] dt$$

avg value of cos over n periods

https://www.falstad.com/fourier/Fourier.html

$$= \frac{1}{T_0}\int_0^{T_0} \sum_{k=1}^{\infty} a_k \frac{1}{2}\left(\cancelto{\phi}{\cos((k+n)\omega_0 t)} + \underbrace{\cos((k-n)\omega_0 t)}_{\substack{=\phi \text{ if } k\neq n \\ =1 \text{ if } k=n}}\right) + b_k \frac{1}{2}\left(\cancelto{\phi}{\cos((k+n)\omega_0 t - 90^\circ)} + \cancelto{\phi}{\cos((k-n)\omega_0 t - 90^\circ)}\right)$$

(k+n) period average

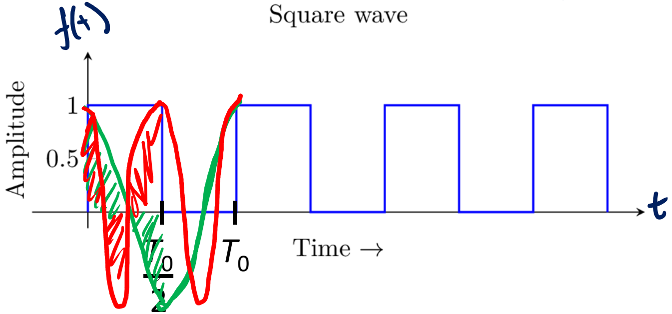
$$\frac{1}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\,dt = \begin{cases} \phi & \text{if } k \neq n \\ \frac{a_n}{2} & \text{if } k = n \end{cases}$$

so,

$$\boxed{a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\,dt}$$

$$\boxed{b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\omega_0 t)\,dt}$$

Example Calculation



period : $T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$

$\omega_0 T_0 = 2\pi$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\,dt$$

$$= \frac{1}{T_0}\left[\int_0^{T_0/2} \underbrace{f(t)}_{=1 \text{ on } [0, T_0/2]}dt + \cancel{\int_{T_0/2}^{T_0} \underbrace{f(t)}_{=\phi \text{ on } [T_0/2, T_0]} dt}^{\phi}\right] = \boxed{a_0 = \frac{1}{2}}$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_0^{T_0/2}(1)\cdot\cos(n\omega_0 t)\,dt = \frac{2}{T_0}\left[\frac{1}{n\omega_0}\sin(n\omega_0 t)\right]\Bigg|_0^{T_0/2} = \underbrace{\frac{2}{T_0 n\omega_0}}_{\left(\frac{1}{n\pi}\right)}\left[\cancel{\underbrace{\sin\left(n\omega_0\frac{T_0}{2}\right)}_{\sin(n\pi)}}^{\phi} - \cancel{\sin(0)}^{\phi}\right]$$

$$\boxed{a_n = \phi \quad \forall n}$$

$$b_n = \frac{2}{T_0}\int_0^{T_0/2}(1)\sin(n\omega_0 t)\,dt = \frac{2}{T_0}\frac{1}{n\omega_0}\left[-\cos(n\omega_0 t)\right]\Bigg|_0^{T_0/2} = \frac{-1}{n\pi}\left[\underbrace{\cos(n\pi)}_{\pm 1} - \underbrace{\cos(0)}_{=+1}\right]$$

$$\boxed{b_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{2}{n\pi} & \text{for } n \text{ odd}\end{cases}}$$