Circuits II

Prof. Daniel Costinett

ECE 202 Lecture 22
March 22, 2024

THE UNIVERSITY OF
TENNESSEE
KNOXVILLE

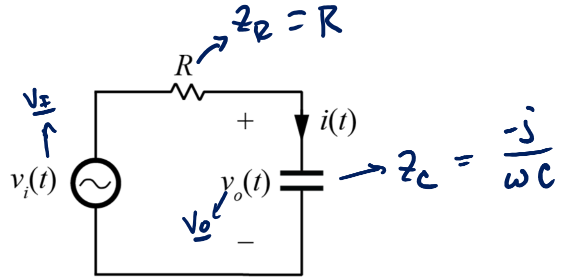
Announcements

- HW 6 posted, due Wed 3/27
  - Half assignment, two problems on Fourier Series
- Experiment 2 due today

Frequency Response

Phasor Analysis:

$$\underline{V_O} = \underline{V_I}\,\frac{Z_C}{Z_C + Z_R} = \underline{V_I}\,\frac{-j/\omega C}{-j/\omega C + R} = \underline{V_I}\left(\frac{1}{1 - j\omega CR}\right)$$



$$\underline{V_O} = \underline{V_I}\left(\frac{1}{1 - j\omega CR}\right) = \underline{V_I}\,H(j\omega)$$

→ Frequency Response

→ tells us, at any ω, how does circuit alter input at the output

$H(j\omega) = \frac{1}{1 - j\omega RC}$

Gain: $|H(j\omega)| = \frac{1}{\sqrt{1^2 + (\omega RC)^2}}$

Phase: $\measuredangle H(j\omega) = 0 - \tan^{-1}\left(\frac{\omega RC}{1}\right)$

$= -\tan^{-1}(\omega RC)$

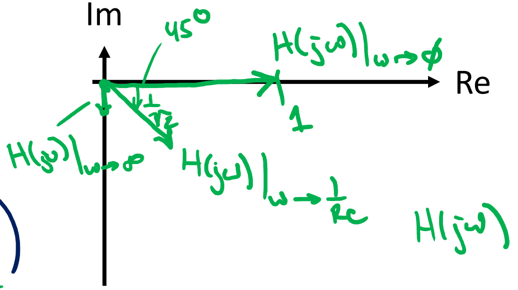
Any LTI circuit has $\underline{V_O} = \underline{V_I}\,H(j\omega)$

In polar form

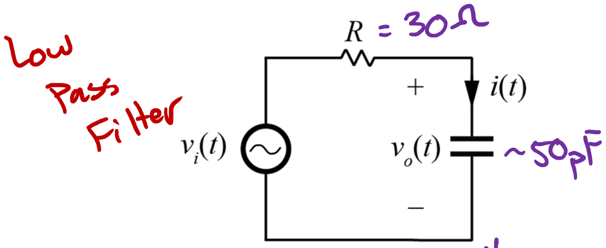
$$V_{OA} \measuredangle \varphi_{VO} = (V_{IA} \measuredangle \varphi_{Vin}) \cdot |H(j\omega)| \measuredangle \varphi(H(j\omega))$$

$$= (V_{IA}\,|H(j\omega)|) \measuredangle \varphi_{Vin} + \varphi(H(j\omega))$$

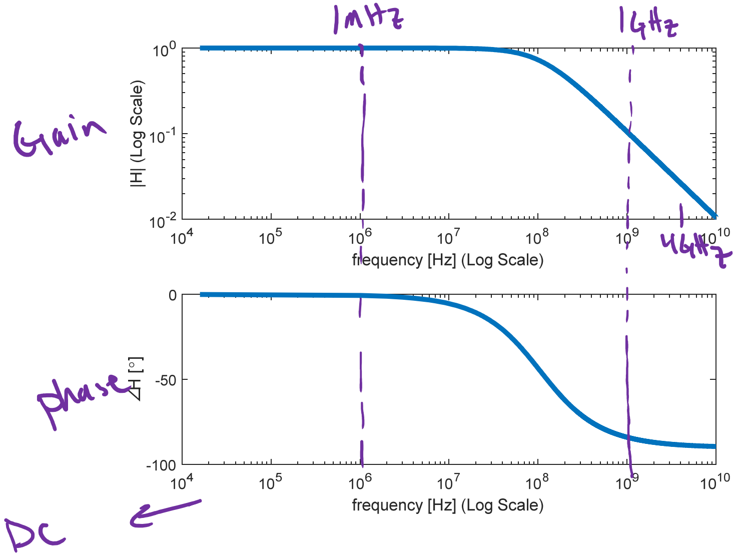
Magnitudes multiply — Phases add

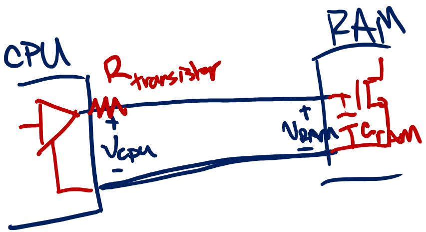


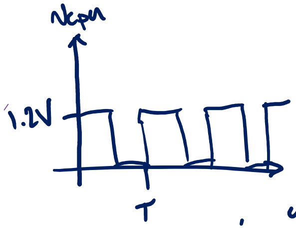
Application: Digital Communication



$H(j\omega) = \frac{V_o}{V_I}$





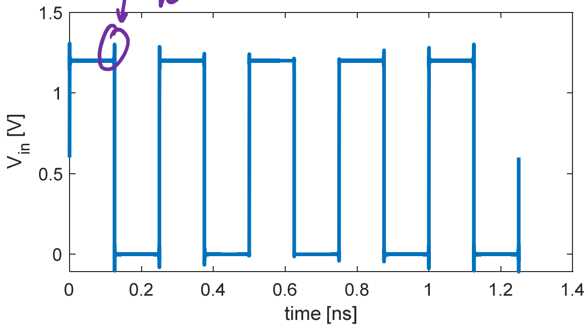


$$\begin{cases} a_0 = 0.6V \\ a_n = \phi \\ b_n = \begin{cases} 0, & n \text{ even} \\ \frac{2}{n\pi}(1.2), & n \text{ odd} \end{cases} \end{cases}$$

$\omega_0 = \frac{2\pi}{T} = 2\pi f, \quad f = 4GHz$

Applying Superposition

need $k > 1000$ to remove



$$v_i(t) = f(t) = a_0 + \sum_{k=1,3,5\ldots}^{\infty} b_k \sin(k\omega_0 t)$$

← first 1000 indices of summation

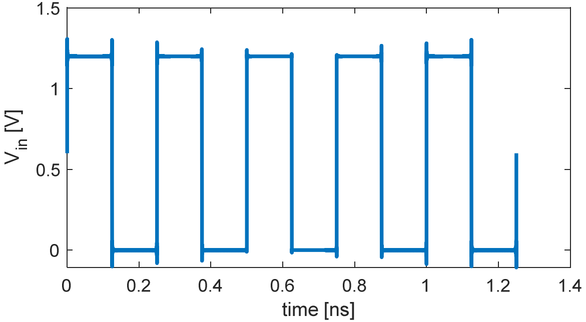
↓

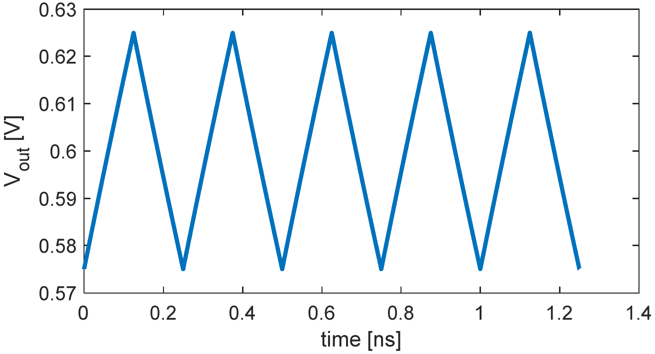
$$H(j\omega) = \frac{1}{1 - j\omega RC}$$

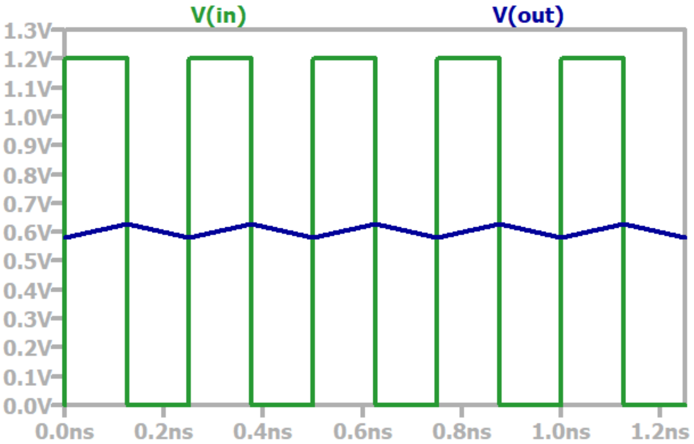
↓

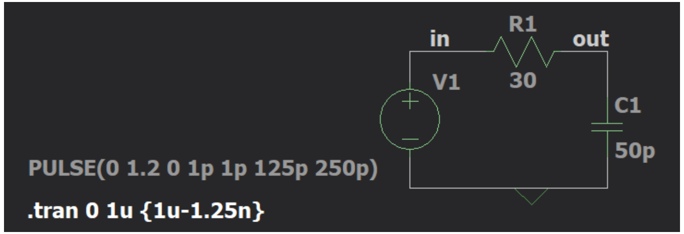
$$v_o(t) = a_0 \left|H(j\omega \to 0)\right| + \sum_{k=1}^{\infty} \left|H(jk\omega_0)\right| b_k \sin\left(k\omega_0 t - \angle H(jk\omega_0)\right)$$

Simulation Verification

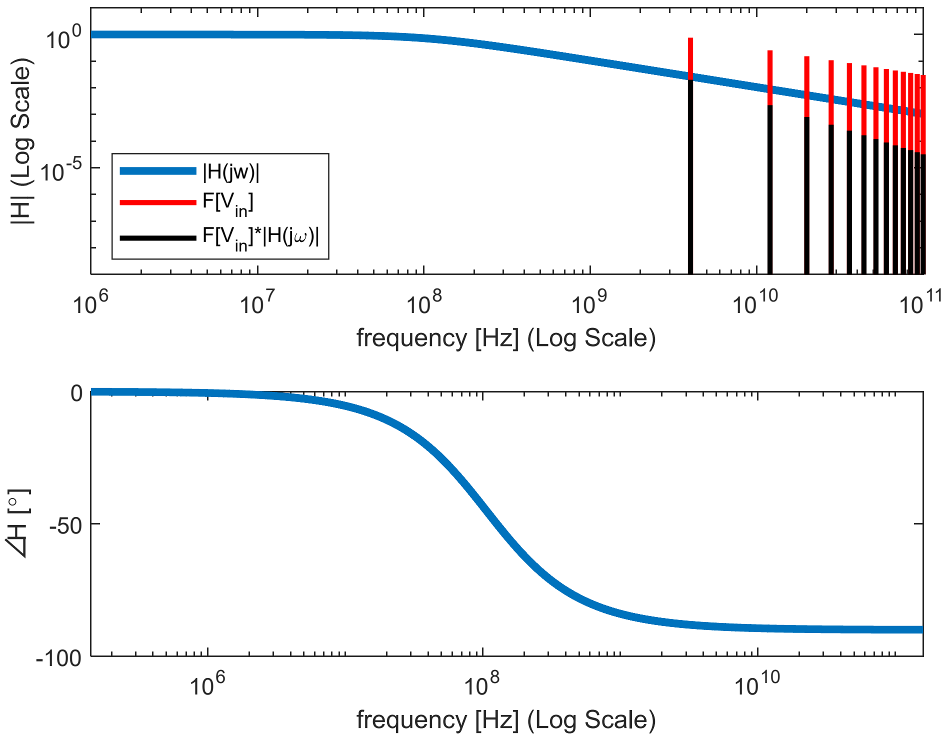




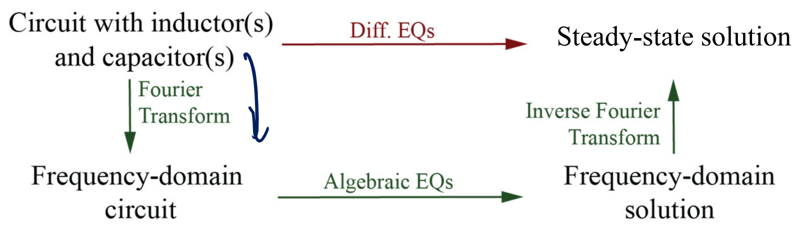




Frequency Domain Interpretation



Fourier Circuit Analysis

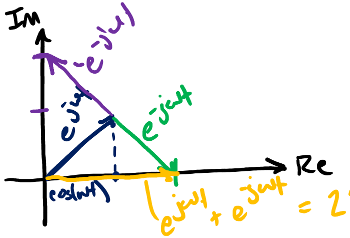


Complex Form of Fourier Series

Euler: $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$

$\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$



$e^{j\omega t} + e^{-j\omega t} = 2\,\mathrm{Re}\{e^{j\omega t}\} = 2\cos(\omega t)$

Plug into Fourier series:

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$= a_0 + \sum_{k=1}^{\infty} \frac{a_k}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + \frac{b_k}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)$$

$$= \underbrace{a_0}_{c_0} + \sum_{k=1}^{\infty} \underbrace{\left(\frac{a_k}{2} - j\frac{b_k}{2}\right)}_{c_k,\ k>1} e^{jk\omega_0 t} + \underbrace{\left(\frac{a_k}{2} + j\frac{b_k}{2}\right)}_{c_k,\ k<1} e^{-jk\omega_0 t}$$

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}} \longrightarrow \boxed{c_k = \frac{1}{T}\int_0^T f(t)\, e^{-jk\omega_0 t}\, dt}$$

$c_k^* = c_{-k}$

Fourier Series Representation

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$f(t)$ can be expressed this way if

1. $f(t)$ is single-valued
2. $\int_{t_0}^{t_0+T_0} |f(t)|\, dt$ exists
3. $f(t)$ had finite discontinuities and max/min per period

$$a_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t) \sin(k\omega_0 t)\, dt$$

Alternate forms

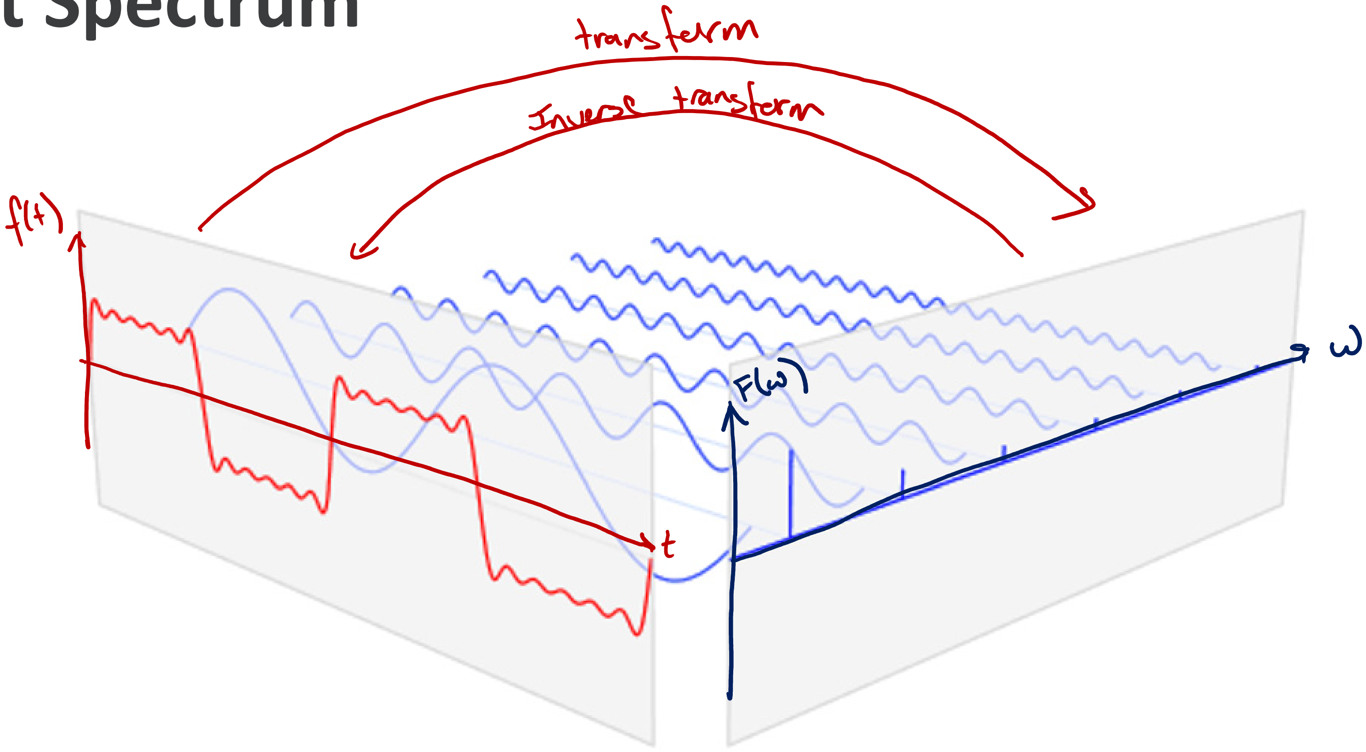
$$f(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \varphi_k) \quad \begin{cases} A_k = \sqrt{{a_k}^2 + {b_k}^2} \\ \varphi_k = \tan^{-1}\left(\frac{b_k}{a_k}\right) \end{cases}$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \begin{cases} c_k = \frac{1}{2}(a_k - jb_k) \\ c_{-k} = \frac{1}{2}(a_k + jb_k) \\ c_0 = a_0 \end{cases}$$

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t} dt$$

Fourier Series & Frequency Domain

Input Spectrum



Fourier Series of a Pulse Train

$D = \frac{\tau}{T}$

$$a_0 = A\frac{\tau}{T}$$

$$b_k = 0$$

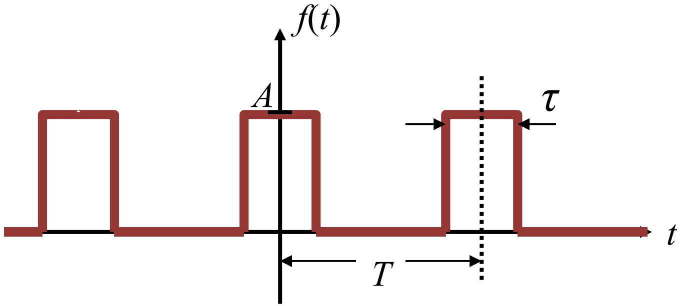
$$a_k = \frac{2A}{k\pi}\sin\left(k\pi\frac{\tau}{T}\right)$$

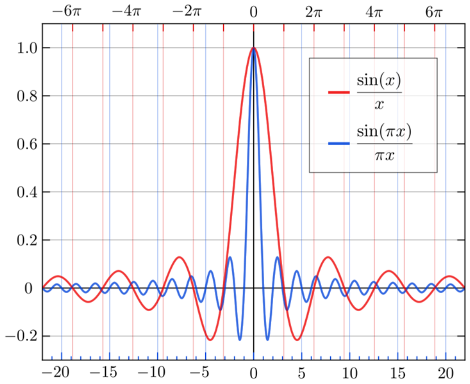
$$c_k = \frac{A}{k\pi}\sin\left(k\pi\frac{\tau}{T}\right)$$

$$a_k = a_0\frac{T}{\tau}\frac{1}{k\pi}\sin\left(k\pi\frac{\tau}{T}\right)$$

$$x = k\pi\frac{\tau}{T}$$

$$a_k = 2a_0\frac{1}{x}\sin(x) = 2a_0\,\mathrm{sinc}(x)$$



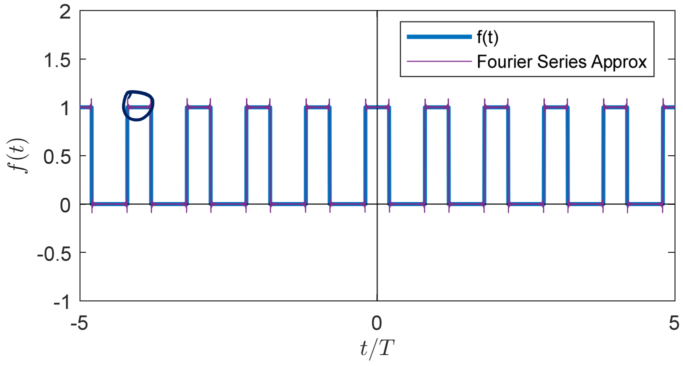


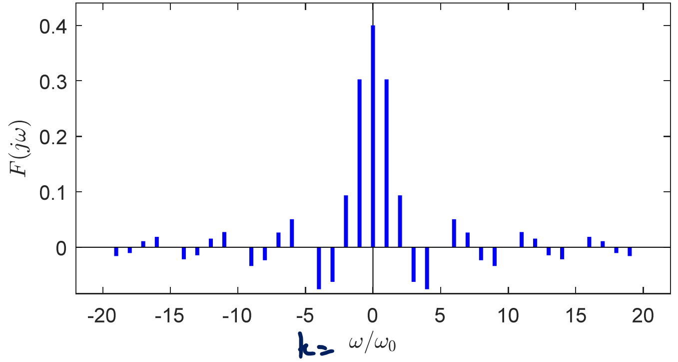
Example Matlab Calculation

$f = 200$ Hz
$T = 5$ ms $= \frac{1}{f}$
$\tau = 2$ ms

Fourier Series Approx

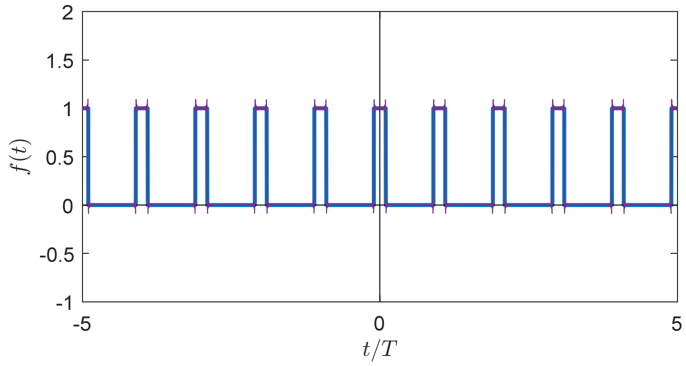
```
f = 200;
A = 1;
tau = 2e-3;

t = linspace(-1/f*5,1/f*5,100000);
a0 = A*tau*f;

sum = a0*(t./t);
kmax = 200;
for k=1:kmax
    ak(k) = 2*A/k/pi*sin(k*pi*D);
    sum = sum + ak(k)*cos(k*w0*t);
end
```

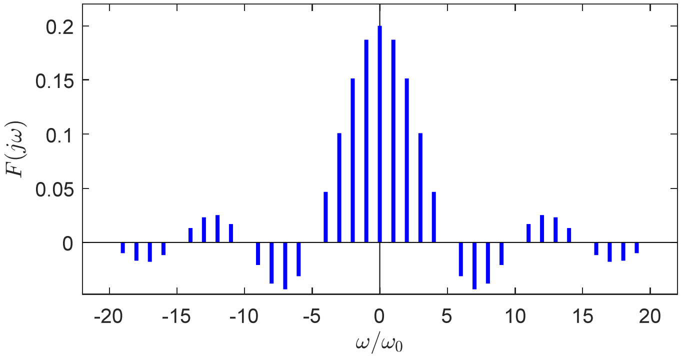




Example Matlab Calculation

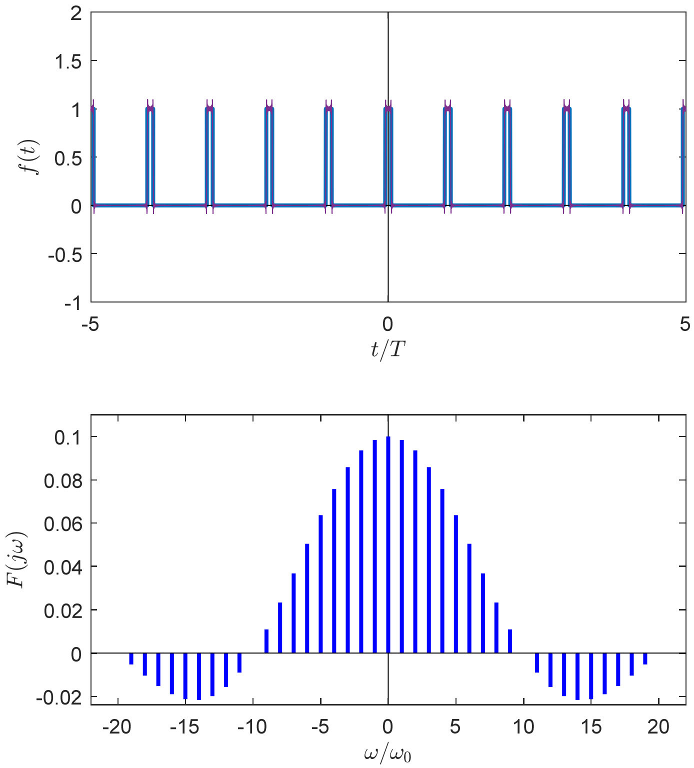
$f = 100$ Hz
$T = 10$ ms
$\tau = 2$ ms





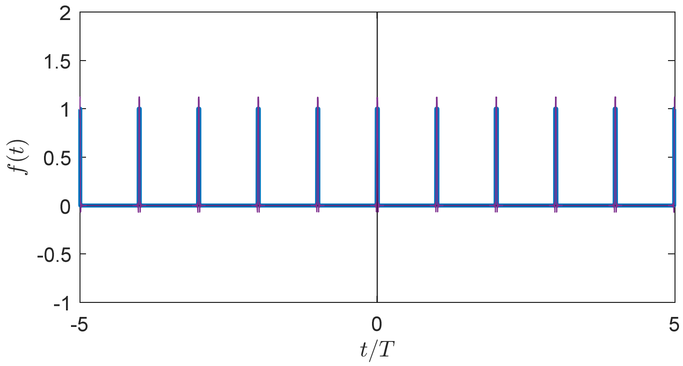
Example Matlab Calculation

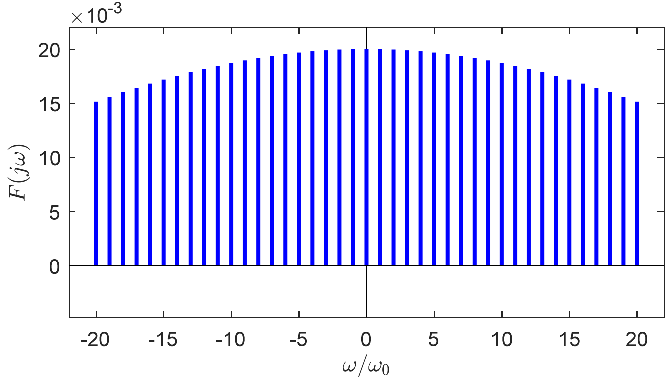
f = 50 Hz
T = 20 ms
τ = 2 ms



Example Matlab Calculation

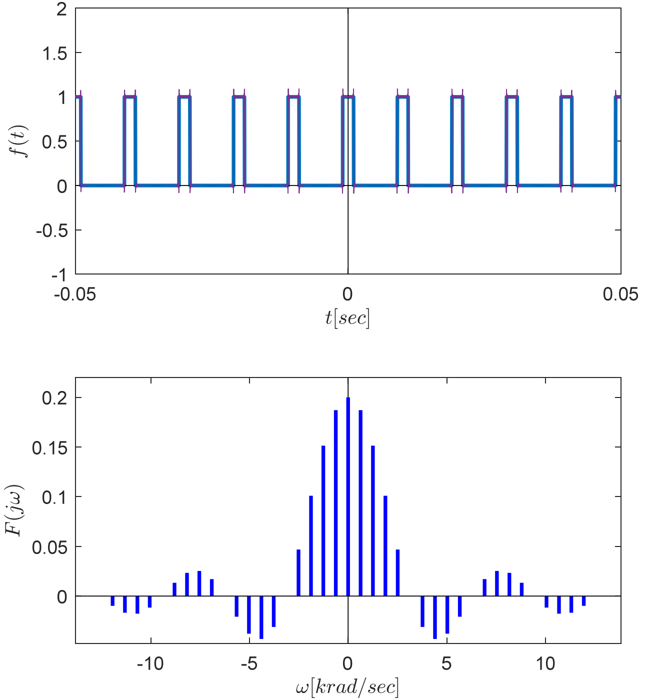
$f = 10$ Hz
$T = 100$ ms
$\tau = 2$ ms



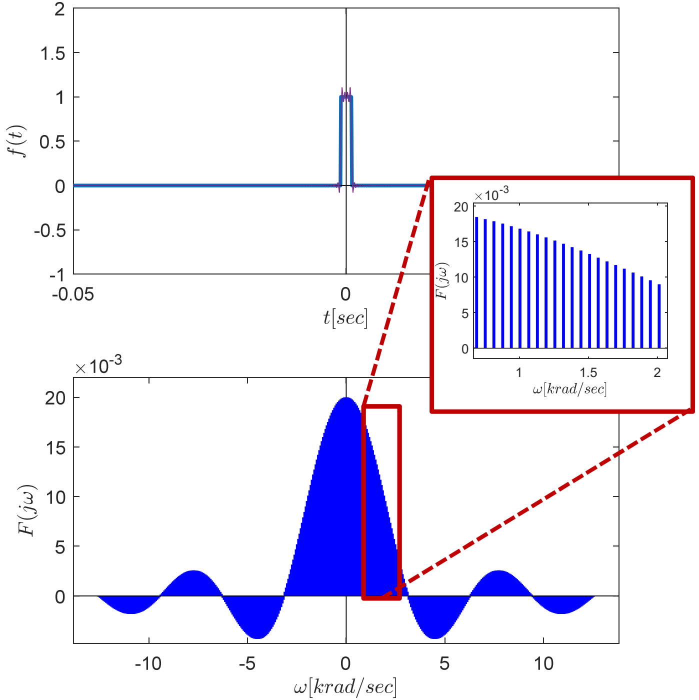


Alternate View

$f = 100$ Hz
$T = 10$ ms
$\tau = 2$ ms



$f = 10$ Hz
$T = 100$ ms
$\tau = 2$ ms



Non-periodic Waveforms: Fourier Transform

Fourier Series → works only for periodic waveforms

Fourier Transform → for non-periodic signals

Idea: treat any non-periodic signal as if it was periodic with $T \to \infty$

Fourier Series: $c_k = \frac{1}{T}\int_0^T f(t)\, e^{-jk\omega_0 t}\, dt$

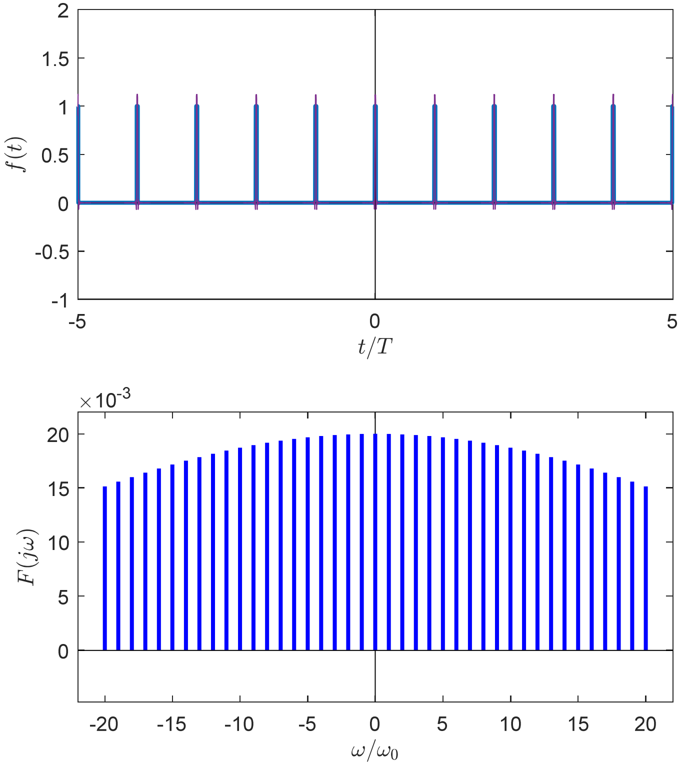
Fourier Transform: $T c_k = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$

Fourier Series: summation $f(t) = \sum_{n=-\infty}^{\infty} c_k\, e^{jk\omega_0 t}$

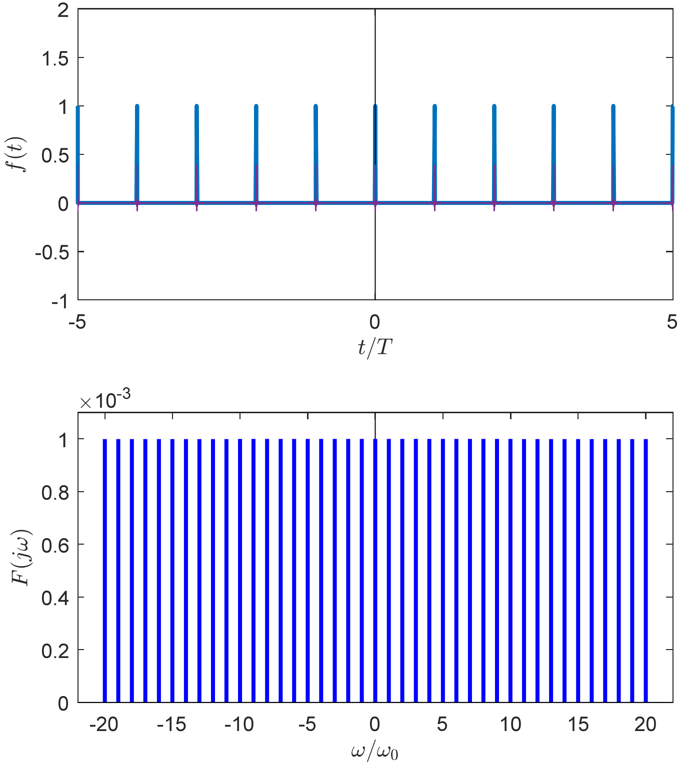
Inverse Fourier Transform: $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, dt$

Fourier Series of Impulse Train

$f = 10$ Hz
$T = 100$ ms
$\tau = 2$ ms



$f = 1000$ Hz
$T = 1$ ms
$\tau = .02$ ms



Applications of Fourier Transform

- Imaging
  - Spectroscopy, x-ray crystallography
  - MRI, CT Scan
- Image analysis
  - Compression
  - Feature extraction
- Signal processing
  - Audio filtering
  - Spike detection
  - Modeling sampled systems (A/D & D/A)
  - Understanding aliasing
  - Speech recognition
- RF Communications
  - AM & FM Encoding

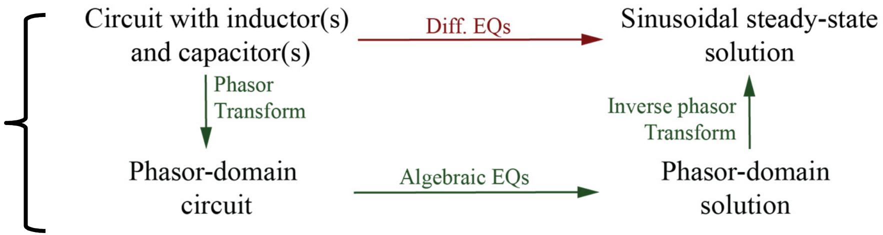
Chapter 14

S-DOMAIN CIRCUIT ANALYSIS

Transform Domains

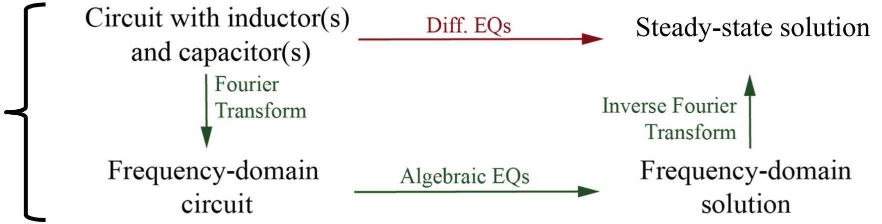
Phase Transform

$$u(t) = A\cos(\omega_0 t + \varphi)$$



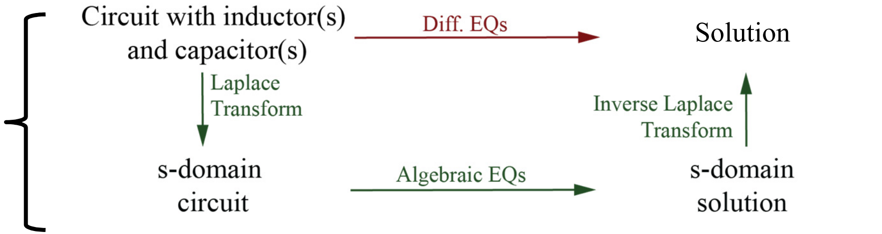
Fourier Transform

$$u(t) = \sum C_n \cos(n\omega_0 t + \varphi_n)$$



Laplace Transform

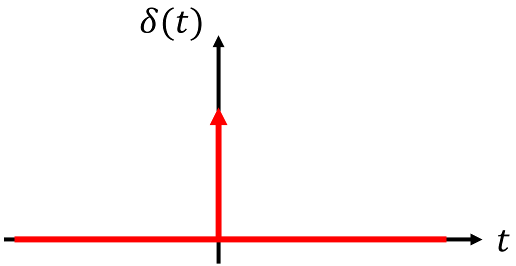
$$u(t) = \sum K_n e^{s_n t}$$



The Laplace Transform

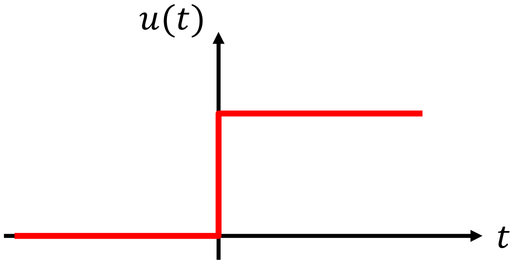
Complex Frequency

Impulse, Step, and Ramp Functions

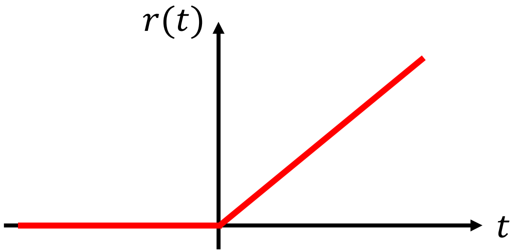


$$\delta(t)\begin{cases}0 & t \neq 0\\ \infty & t = 0\end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$



$$u(t)\begin{cases}0 & t < 0\\ 1 & t > 0\end{cases}$$



$$r(t) = tu(t) = \begin{cases}0 & t \leq 0\\ t & t \geq 0\end{cases}$$

Example Signal Laplace Transforms

TABLE 14.1 Laplace Transform Pairs

$f(t) = \mathscr{L}^{-1}\{\mathbf{F}(\mathbf{s})\}$	$\mathbf{F}(\mathbf{s}) = \mathscr{L}\{f(t)\}$	$f(t) = \mathscr{L}^{-1}\{\mathbf{F}(\mathbf{s})\}$	$\mathbf{F}(\mathbf{s}) = \mathscr{L}\{f(t)\}$
$\delta(t)$	1	$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})\,u(t)$	$\frac{1}{(\mathbf{s} + \alpha)(\mathbf{s} + \beta)}$
$u(t)$	$\frac{1}{\mathbf{s}}$	$\sin \omega t\, u(t)$	$\frac{\omega}{\mathbf{s}^2 + \omega^2}$
$tu(t)$	$\frac{1}{\mathbf{s}^2}$	$\cos \omega t\, u(t)$	$\frac{\mathbf{s}}{\mathbf{s}^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!}u(t),\ n = 1, 2, \ldots$	$\frac{1}{\mathbf{s}^n}$	$\sin(\omega t + \theta)\, u(t)$	$\frac{\mathbf{s}\sin\theta + \omega\cos\theta}{\mathbf{s}^2 + \omega^2}$
$e^{-\alpha t}u(t)$	$\frac{1}{\mathbf{s} + \alpha}$	$\cos(\omega t + \theta)\, u(t)$	$\frac{\mathbf{s}\cos\theta - \omega\sin\theta}{\mathbf{s}^2 + \omega^2}$
$te^{-\alpha t}u(t)$	$\frac{1}{(\mathbf{s} + \alpha)^2}$	$e^{-\alpha t}\sin \omega t\, u(t)$	$\frac{\omega}{(\mathbf{s} + \alpha)^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t),\ n = 1,2,\ldots$	$\frac{1}{(\mathbf{s} + \alpha)^n}$	$e^{-\alpha t}\cos \omega t\, u(t)$	$\frac{\mathbf{s} + \alpha}{(\mathbf{s} + \alpha)^2 + \omega^2}$

Properties of the Laplace Transform

TABLE 14.2 Laplace Transform Operations

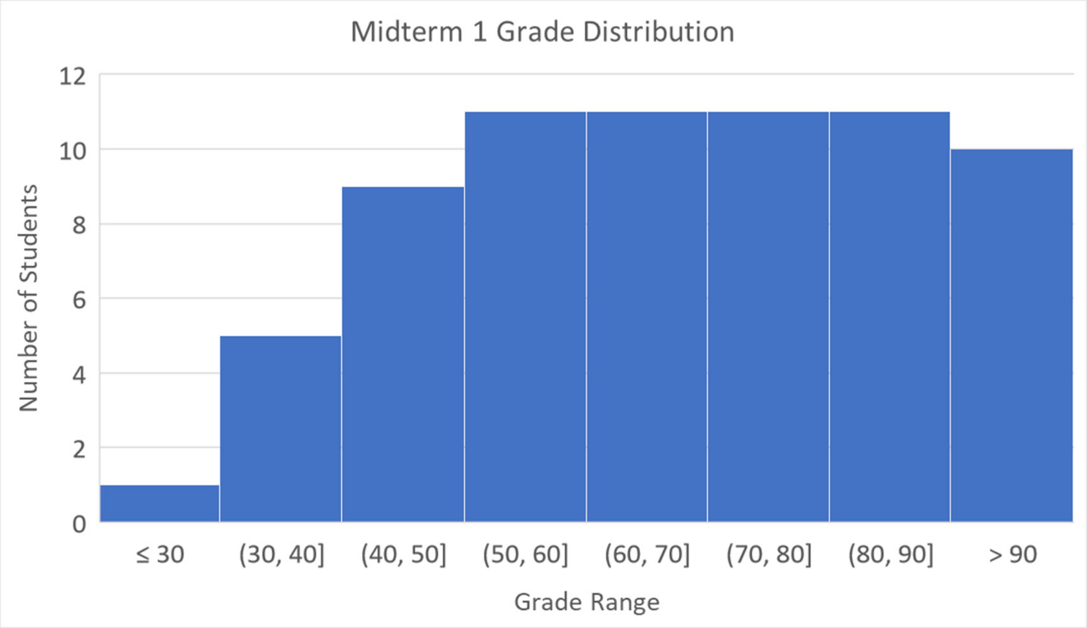
Operation	f(t)	F(s)
Addition	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(s) \pm \mathbf{F}_2(s)$
Scalar multiplication	$kf(t)$	$k\mathbf{F}(s)$
Time differentiation	$\frac{df}{dt}$	$s\mathbf{F}(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2\mathbf{F}(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3\mathbf{F}(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
Time integration	$\int_{0^-}^{t} f(t)\,dt$	$\frac{1}{s}\mathbf{F}(s)$
	$\int_{-\infty}^{t} f(t)\,dt$	$\frac{1}{s}F(s) + \frac{1}{s}\int_{-\infty}^{0^-} f(t)\,dt$
Convolution	$f_1(t) * f_2(t)$	$\mathbf{F}_1(s)\mathbf{F}_2(s)$
Time shift	$f(t-a)u(t-a),\ a \geq 0$	$e^{-as}\mathbf{F}(s)$
Frequency shift	$f(t)e^{-at}$	$\mathbf{F}(s+a)$
Frequency differentiation	$tf(t)$	$-\frac{d\mathbf{F}(s)}{ds}$
Frequency integration	$\frac{f(t)}{t}$	$\int_{s}^{\infty} \mathbf{F}(s)\,ds$
Scaling	$f(at),\ a \geq 0$	$\frac{1}{a}\mathbf{F}\left(\frac{s}{a}\right)$
Initial value	$f(0^+)$	$\lim_{s\to\infty} s\mathbf{F}(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} s\mathbf{F}(s)$, all poles of $s\mathbf{F}(s)$ in LHP
Time periodicity	$f(t) = f(t+nT)$, $n = 1, 2, \ldots$	$\frac{1}{1-e^{-T_s}}\mathbf{F}_1(s)$, where $\mathbf{F}_1(s) = \int_{0^-}^{T} f(t)e^{-st}\,dt$

Circuit Laplace Transform

Differential Equation Laplace Transform

Transfer Functions

Midterm Exam

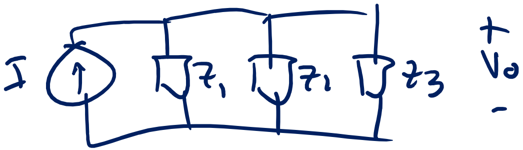


- Mean: 67.4%
- Median: 68.2%

- Grading

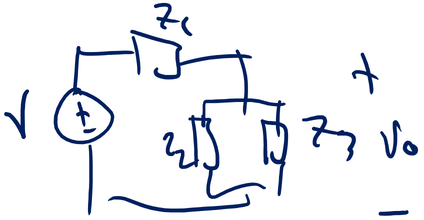
Trends

- Direct Solutions
  - Any source + 3-4 2-terminal impedances



$$V_o = I\,(Z_1 \| Z_2 \| Z_3)$$

$$\frac{1}{\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}}$$



$$V_o = \frac{Z_2 \| Z_3}{Z_2 \| Z_3 + Z_1}\,V$$

- SI prefixes