Complex Form of Fourier Series

Fourier Series Representation

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$f(t)$ can be expressed this way if

1. $f(t)$ is single-valued
2. $\int_{t_0}^{t_0+T_0} |f(t)|\, dt$ exists
3. $f(t)$ had finite discontinuities and max/min per period

$$a_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t) \sin(k\omega_0 t)\, dt$$

Alternate forms

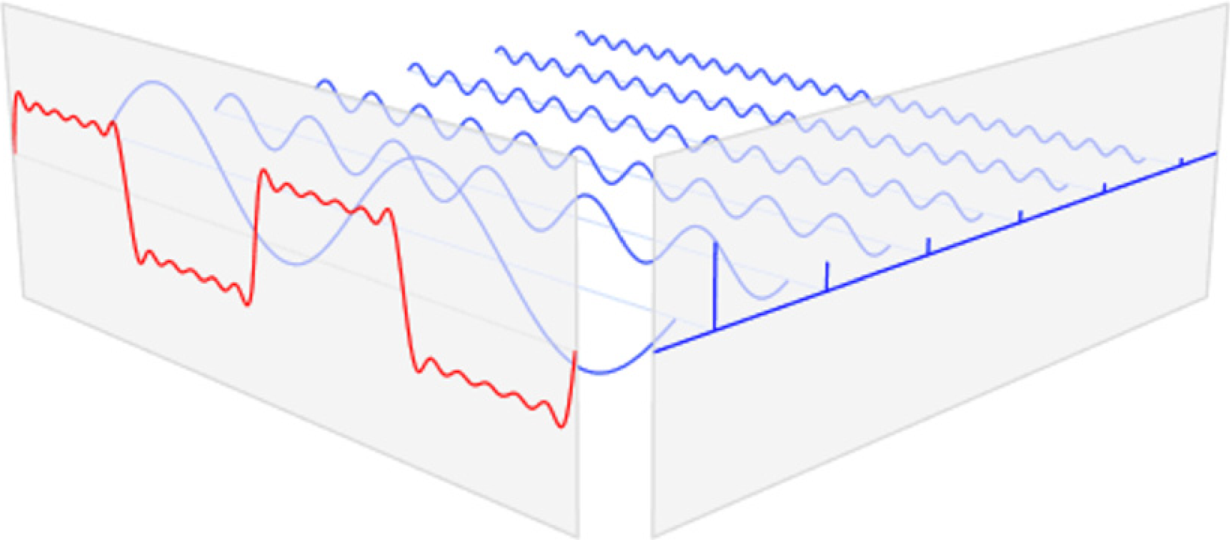
$$f(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \varphi_k) \quad \begin{cases} A_k = \sqrt{{a_k}^2 + {b_k}^2} \\ \varphi_k = \tan^{-1}\left(\frac{b_k}{a_k}\right) \end{cases}$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \begin{cases} c_k = \frac{1}{2}(a_k - jb_k) \\ c_{-k} = \frac{1}{2}(a_k + jb_k) \\ c_0 = a_0 \end{cases}$$

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t} dt$$

Fourier Series & Frequency Domain

Input Spectrum



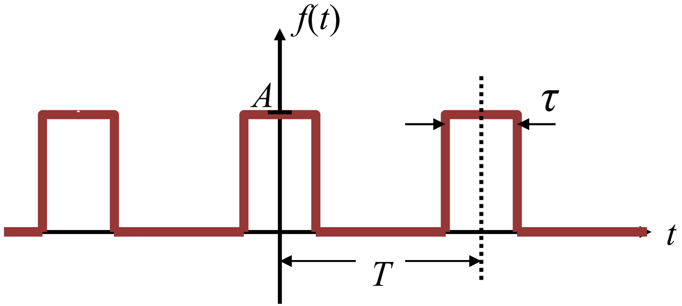
Fourier Series of a Pulse Train

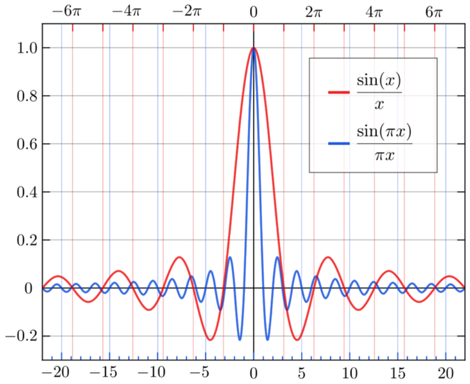
$$a_0 = A\frac{\tau}{T}$$

$$b_k = 0$$

$$a_k = \frac{2A}{k\pi}\sin\left(k\pi\frac{\tau}{T}\right)$$

$$c_k = \frac{A}{k\pi}\sin\left(k\pi\frac{\tau}{T}\right)$$



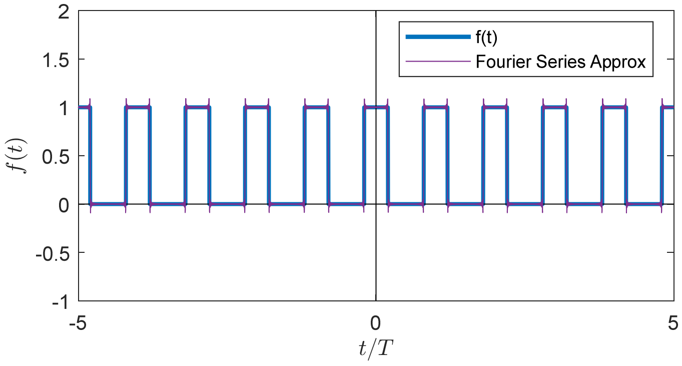


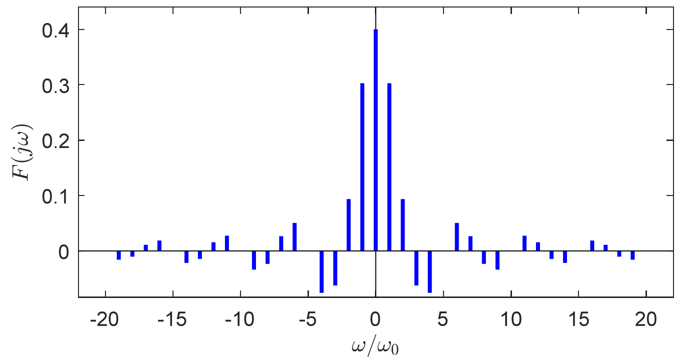
Example Matlab Calculation

$f = 200$ Hz
$T = 5$ ms
$\tau = 2$ ms

Fourier Series Approx

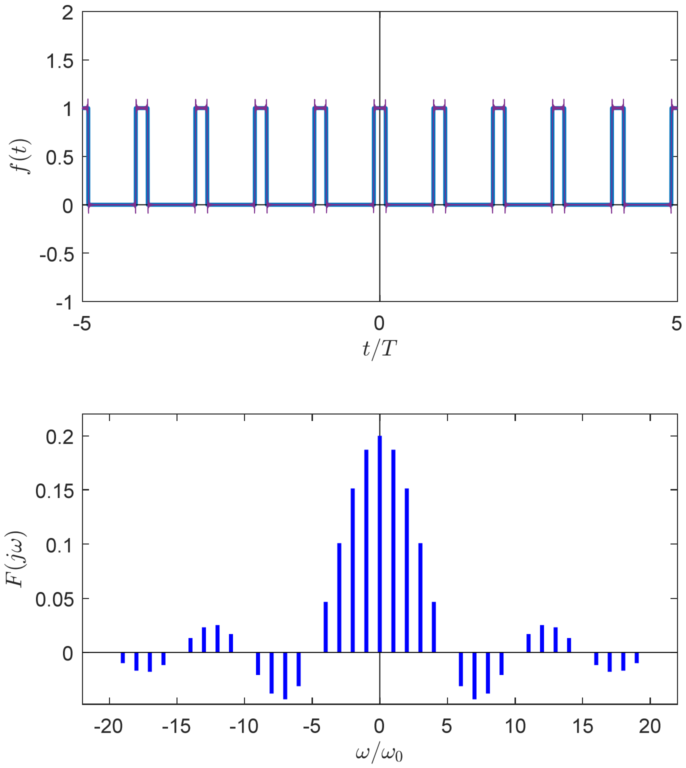
```
f = 200;
A = 1;
tau = 2e-3;

t = linspace(-1/f*5,1/f*5,100000);
a0 = A*tau*f;

sum = a0*(t./t);
kmax = 200;
for k=1:kmax
    ak(k) = 2*A/k/pi*sin(k*pi*D);
    sum = sum + ak(k)*cos(k*w0*t);
end
```





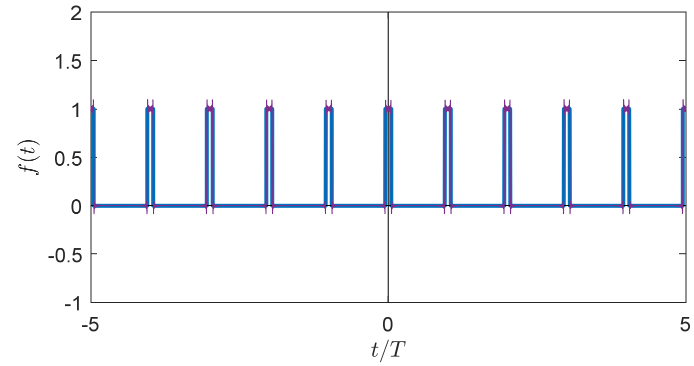
Example Matlab Calculation

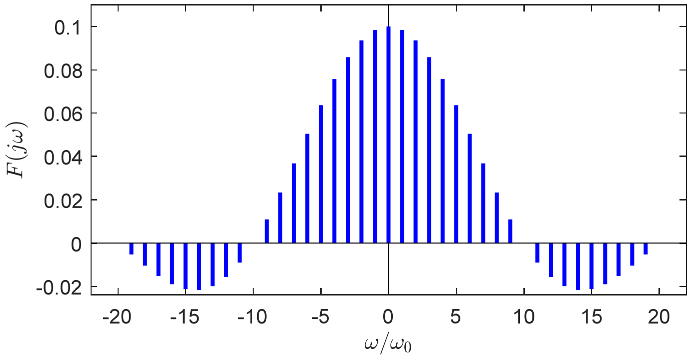
f = 100 Hz
T = 10 ms
τ = 2 ms



Example Matlab Calculation

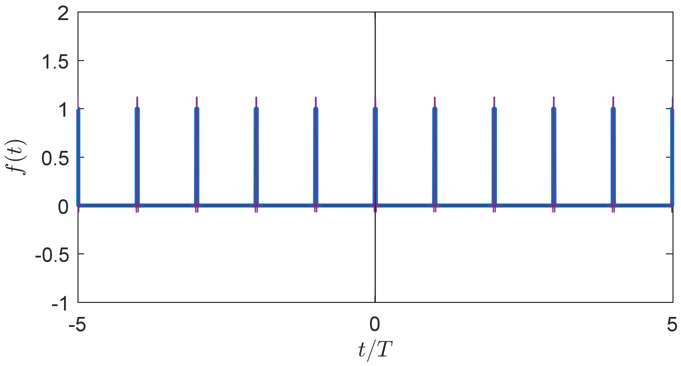
$f = 50$ Hz
$T = 20$ ms
$\tau = 2$ ms

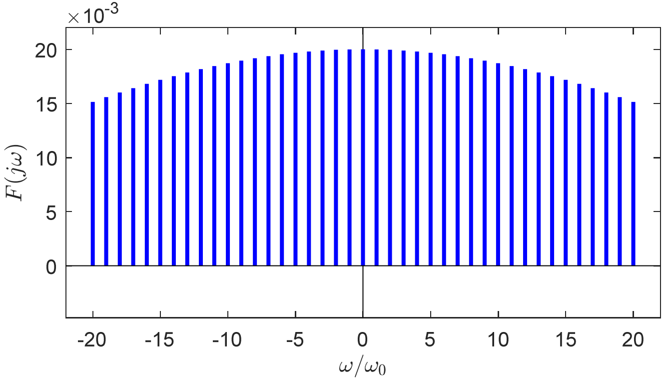




Example Matlab Calculation

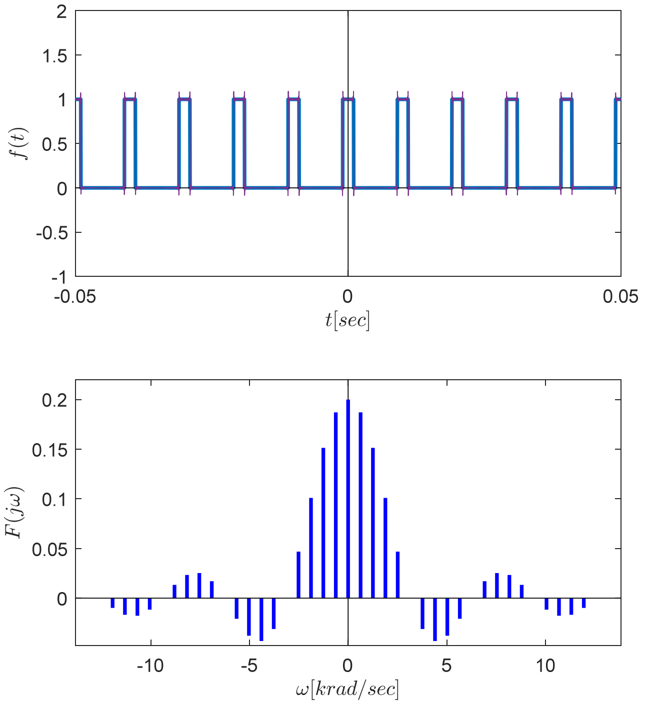
$f = 10$ Hz
$T = 100$ ms
$\tau = 2$ ms



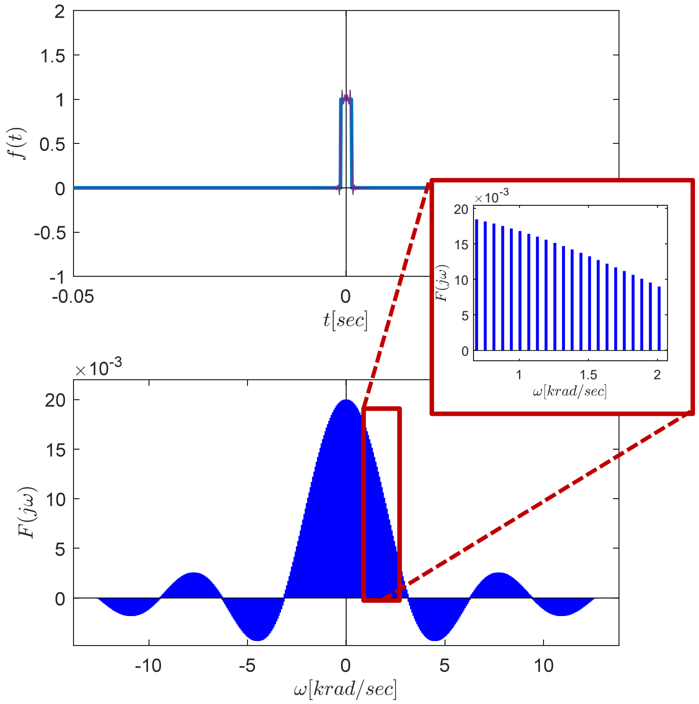


Alternate View

f = 100 Hz
T = 10 ms
τ = 2 ms



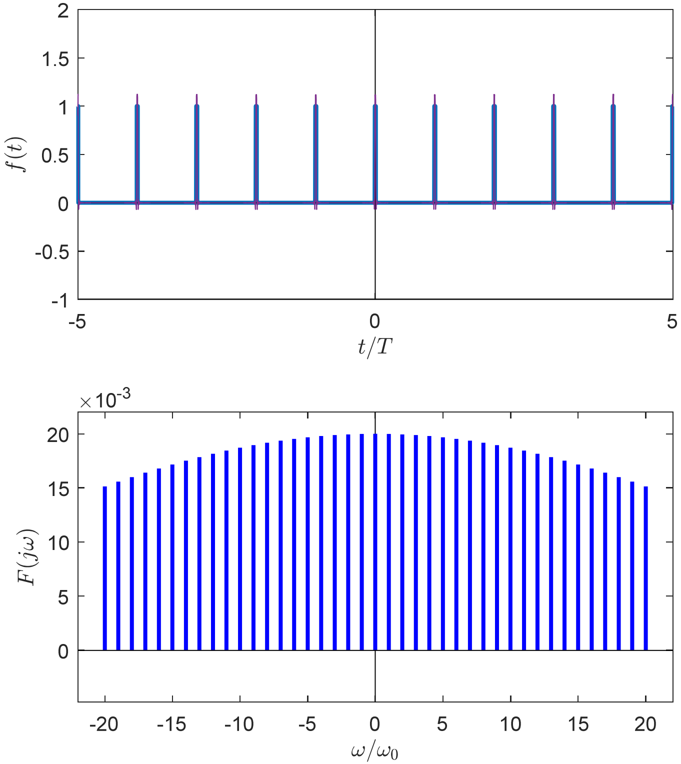
f = 10 Hz
T = 100 ms
τ = 2 ms

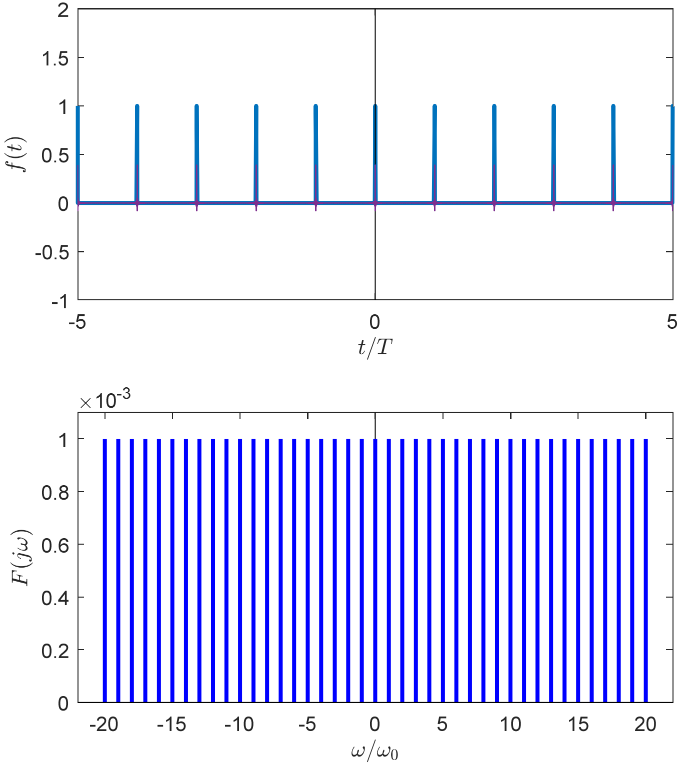


Fourier Series of Impulse Train

$f = 10$ Hz
$T = 100$ ms
$\tau = 2$ ms

$f = 1000$ Hz
$T = 1$ ms
$\tau = .02$ ms





Non-periodic Waveforms: Fourier Transform

Applications of Fourier Transform

- Imaging
  - Spectroscopy, x-ray crystallography
  - MRI, CT Scan
- Image analysis
  - Compression
  - Feature extraction
- Signal processing
  - Audio filtering
  - Spike detection
  - Modeling sampled systems (A/D & D/A)
  - Understanding aliasing
  - Speech recognition
- RF Communications
  - AM & FM Encoding