Transforms Visualized

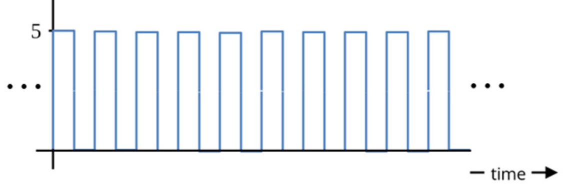
Fourier Series

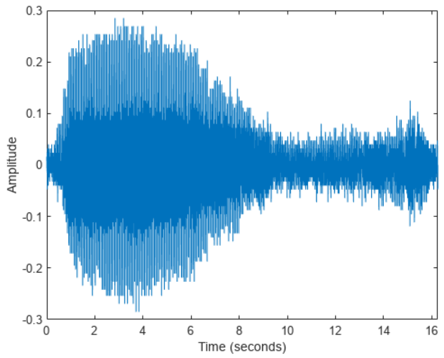
Fourier Transform

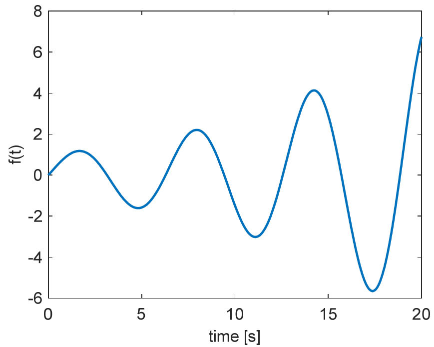
Laplace Transform

Time Domain

$$x(t) = \begin{cases} 5 & nT < t \le (n+1)\dfrac{T}{2} \\ 0 & otherwise \end{cases}$$

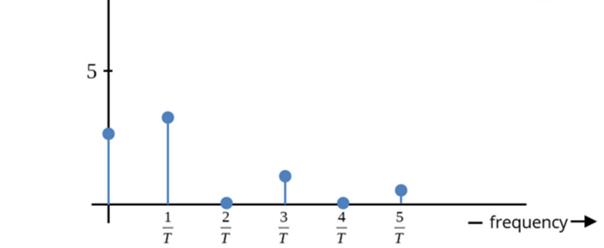


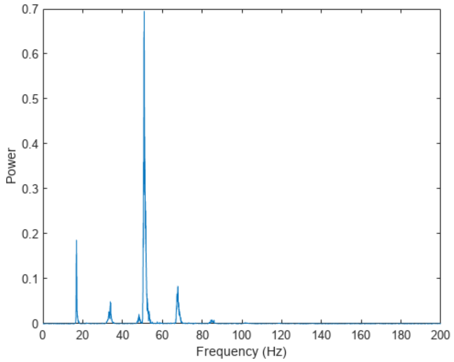


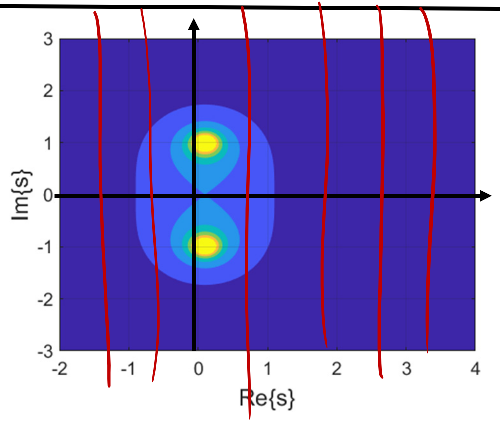


Frequency Domain

$$X(f) = 2.5\,\delta(f) + 3.18\,\delta\left(f - \frac{1}{T}\right) + 1.06\,\delta\left(f - \frac{3}{T}\right) + \ \ldots$$





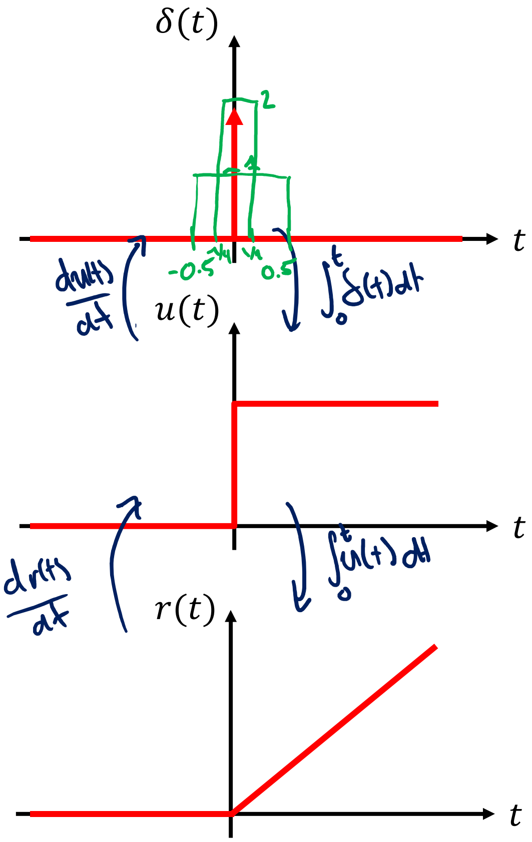


Impulse, Step, and Ramp Functions

Impulse

step

ramp



$$\delta(t)\begin{cases}0 & t \neq 0\\ \infty & t = 0\end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$

$$u(t)\begin{cases}0 & t < 0\\ 1 & t > 0\end{cases}$$

$$r(t) = tu(t) = \begin{cases}0 & t \leq 0\\ t & t \geq 0\end{cases}$$

Sifting Property of Impulse Function $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

so

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

Example Signal Laplace Transforms

$f(t) = u(t)$

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} e^{-st} u(t)\,dt = \int_0^{\infty} e^{-st}\,dt$$

$$= \left[\frac{-1}{s} e^{-st}\right]\Bigg|_{t=0}^{t=\infty} = \left[0 - \left(\frac{-1}{s}\right)\right]$$

$$\boxed{F(s) = \mathcal{L}\{u(t)\} = \frac{1}{s} \quad \text{if } \operatorname{Re}\{s\} > 0}$$

Region of convergence for $\mathcal{L}\{u(t)\} \rightarrow \operatorname{Re}\{s\} > 0$

$s = \sigma + j\omega \rightarrow \sigma > 0$

$f(t) = e^{-at} u(t)$

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} e^{-at} u(t)\,dt = \int_0^{\infty} e^{-(s+a)t}\,dt$$

$$\boxed{\mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a} \quad \text{if } \operatorname{Re}\{s+a\} > 0}$$

Generalize: $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

(where $F(s) = \mathcal{L}\{f(t)\}$)

$f(t) = r(t) = t\,u(t)$

$$\mathcal{L}\{r(t)\} = \int_{0^-}^{\infty} e^{-st}\, t\, u(t)\, dt = \int_0^{\infty} \underbrace{e^{-st}}_{\frac{dv}{dt}}\, \underbrace{t}_{u}\, dt =$$

Apply Integration by parts

$$\int_0^{\infty} \frac{dv}{dt} u \, dt = (uv)\Big|_0^{\infty} - \int_0^{\infty} \frac{du}{dt} v \, dt$$

$$= \left[\underbrace{t}_{u} \cancelto{0}{\underbrace{\left(\frac{-1}{s}\right) e^{-st}}_{v}} \right]\Bigg|_0^{\infty} - \int_0^{\infty} \underbrace{(1)}_{\frac{du}{dt}} \underbrace{\frac{-1}{s} e^{-st}}_{v}\, dt$$

$$= \left[-\frac{1}{s^2} e^{-st} \right]\Bigg|_0^{\infty} = \left[0 - \left(-\frac{1}{s^2} \right) \right]$$

$$\boxed{\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}, \quad \text{if } \operatorname{Re}\{s\} > 0}$$

TABLE 14.1 Laplace Transform Pairs

$f(t) = \mathscr{L}^{-1}\{\mathbf{F}(\mathbf{s})\}$	$\mathbf{F}(\mathbf{s}) = \mathscr{L}\{f(t)\}$	$f(t) = \mathscr{L}^{-1}\{\mathbf{F}(\mathbf{s})\}$	$\mathbf{F}(\mathbf{s}) = \mathscr{L}\{f(t)\}$
$\delta(t)$	1	$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})\,u(t)$	$\frac{1}{(\mathbf{s} + \alpha)(\mathbf{s} + \beta)}$
$u(t)$	$\frac{1}{\mathbf{s}}$	$\sin \omega t\, u(t)$	$\frac{\omega}{\mathbf{s}^2 + \omega^2}$
$tu(t)$	$\frac{1}{\mathbf{s}^2}$	$\cos \omega t\, u(t)$	$\frac{\mathbf{s}}{\mathbf{s}^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!}u(t), n = 1, 2, \ldots$	$\frac{1}{\mathbf{s}^n}$	$\sin(\omega t + \theta)\, u(t)$	$\frac{\mathbf{s}\sin\theta + \omega\cos\theta}{\mathbf{s}^2 + \omega^2}$
$e^{-\alpha t}u(t)$	$\frac{1}{\mathbf{s} + \alpha}$	$\cos(\omega t + \theta)\, u(t)$	$\frac{\mathbf{s}\cos\theta - \omega\sin\theta}{\mathbf{s}^2 + \omega^2}$
$te^{-\alpha t}u(t)$	$\frac{1}{(\mathbf{s} + \alpha)^2}$	$e^{-\alpha t}\sin \omega t\, u(t)$	$\frac{\omega}{(\mathbf{s} + \alpha)^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t), n = 1,2,\ldots$	$\frac{1}{(\mathbf{s} + \alpha)^n}$	$e^{-\alpha t}\cos \omega t\, u(t)$	$\frac{\mathbf{s} + \alpha}{(\mathbf{s} + \alpha)^2 + \omega^2}$

Properties of the Laplace Transform

1. Uniqueness: if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\} = f(t)$

2. Linearity: $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} = F(s) + G(s)$

$$\mathcal{L}\{\alpha f(t)\} = \alpha F(s)$$

3. Differentiation:

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^{\infty} \underset{u}{e^{-st}} \underset{\frac{dv}{dt}}{\frac{df}{dt}}\,dt = \left(e^{-st} f(t)\right)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} (-s) e^{-st} f(t)\,dt$$

$$= \left(0 - f(0^-)\right) + s\underbrace{\int_{0^-}^{\infty} e^{-st} f(t)\,dt}_{F(s)}$$

$$\boxed{\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)}$$

Differentiation can be applied recursively

$$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s\left[sF(s) - f(0^-)\right] - f'(0^-) = s^2 F(s) - s f(0^-) - f'(0)$$

Integration

$$\mathcal{L}\left\{\int_{0^-}^{t} f(\tau)\,d\tau\right\} = \int_{0^-}^{\infty} \underbrace{e^{-st}}_{\frac{dv}{dt}} \underbrace{\left(\int_{0}^{t} f(\tau)\,d\tau\right)}_{u} dt$$

$$= \left[\left(\int_{0}^{t} f(\tau)\,d\tau\right)\left(\frac{-1}{s}e^{-st}\right)\right]\Bigg|_{0}^{\infty} - \int_{0^-}^{\infty}\left(\frac{-1}{s}\right)e^{-st} f(t)\,dt$$

$$\xrightarrow{} 0$$

$$= \frac{1}{s}\int_{0^-}^{\infty} e^{-st} f(t)\,dt$$

$$\boxed{\mathcal{L}\left\{\int_{0}^{t} f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)}$$

Initial and Final Value Theorems

Initial Value Theorem

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^{\infty} e^{-st}\frac{df}{dt}\,dt = sF(s) - f(0^-)$$

$$\lim_{s\to\infty}\left[\int_{0^-}^{0^+} \underbrace{e^{-st}}_{=1 \,@\, t=0}\frac{df}{dt}\,dt + \cancel{\int_{0^+}^{\infty} e^{-st}\frac{df}{dt}\,dt}\right] = \lim_{s\to\infty}\left[sF(s) - f(0^-)\right]$$

$$= f(0^+) - \cancel{f(0^-)} = \lim_{s\to\infty}\left[sF(s)\right] - \cancel{f(0^-)}$$

$$\boxed{= \lim_{t\to 0^+} f(t) = \lim_{s\to\infty}\left[sF(s)\right]}$$