Laplace Transform of Diff EQs

N^{th} order circuit with sinusoidal input described by ($M \leq N$ for causality)

$$b_N \frac{d^N}{dt^N} v_o(t) + \cdots + b_1 \frac{d}{dt} v_o(t) + b_0 v_o(t) = a_M \frac{d^M}{dt^M} v_i(t) + \cdots + a_1 \frac{d}{dt} v_i(t) + a_0 v_i(t)$$

$$\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_o(t) = \sum_{i=0}^{M} a_i \frac{d^i}{dt^i} v_i(t)$$

Then the Laplace transform of the circuit, neglecting initial conditions, is

$$\mathcal{L}\left\{\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_o(t)\right\} = \mathcal{L}\left\{\sum_{i=0}^{M} a_i \frac{d^i}{dt^i} v_i(t)\right\}$$

Neglect Initial conditions

$$\sum_{i=0}^{N} b_i s^i V_o(s) = \sum_{i=0}^{M} a_i s^i V_i(s)$$

if we replace $s \to j\omega$ in $H(s)$ we get $H(j\omega)$, the frequency response

Rearranging:

Transfer function

$$\boxed{\frac{V_o(s)}{V_i(s)} = H(s) = \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i}}$$

Laplace Circuit Solution Algorithm

1. Transform all sources, signals into Laplace Domain
2. Transform circuit components (including initial conditions) into Laplace Domain
3. Solve the circuit using 201 techniques

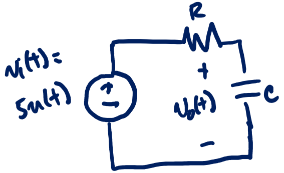
if multiple inputs

if any initial conditions

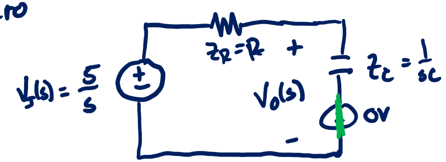
$$\boxed{V_o(s) = H(s)V_i(s) = \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i} V_i(s) + H_2(s)V_{i_2}(s)} + H_3(s)V_{IC1}(s) + \ldots$$

4. Inverse Laplace Transform to get back to time domain

Example Laplace Circuit Analysis



$v_o(t)$ @ $t \le 0$ is zero



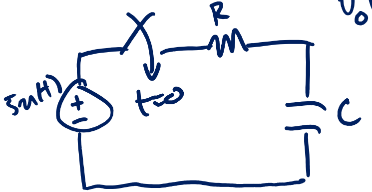
$v_o(t) = \left(5 - 5e^{-\frac{1}{RC}t}\right)u(t)$

$$V_o(s) = V_i(s)\frac{Z_C}{Z_C + Z_R} = \frac{5}{s}\frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{5}{s}\frac{\frac{1}{RC}}{\frac{1}{RC} + s}$$

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{\frac{1}{RC}+s}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{s} + \frac{-5}{\frac{1}{RC}+s}\right\}$$

PFE (more review coming)

$$v(t) = \mathcal{L}^{-1}\left\{\frac{5}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-5}{\frac{1}{RC}+s}\right\} = \boxed{5u(t) - 5e^{-\frac{1}{RC}t}u(t)}$$



$v_o(t) = 1V$ for $t \le 0$

for $t > 0$

$-Cv(0^-)$
$= -C(1V) = -C$

$$V_o(s) = \frac{5}{s}\left(\frac{\frac{1}{sC}}{R + \frac{1}{sC}}\right) + -(-C)\left(R \parallel \frac{1}{sC}\right) \longrightarrow C\left(\frac{1}{\frac{1}{R} + sC}\right) = \cancel{C}\frac{\frac{1}{\cancel{C}}}{\frac{1}{RC} + s} = \frac{1}{\frac{1}{RC} + s}$$

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = 5u(t) - 5e^{-\frac{1}{RC}t}u(t) + e^{-\frac{1}{RC}t}u(t)$$

Inverse Transforms

1. solve laplace domain circuit (for each input/IC source) to get some $V_o(s) = H(s)V_i(s)$

Transfer function

circuit solution

$\mathcal{L}\{v_i(t)\}$

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = \mathcal{L}^{-1}\{H(s)V_i(s)\}$$

this will look like $v_o(t) = \mathcal{L}^{-1}\left\{\frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}\right\}$

some ratio of polynomials of s

Usually, we'll need to factor $V_o(s)$ & do PFE

ex/ $V_o(s) = \frac{10}{s^2+4s+4} = \frac{10}{(s+2)(s+2)} = \frac{10}{(s+2)^2} \rightarrow v_o(t) = \mathcal{L}^{-1}\left\{\frac{10}{(s+2)^2}\right\} = 10te^{-2t}u(t)$

Factor 2nd order polynomial w/ quadratic formula

$as^2 + bs + c = \phi \rightarrow r_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$a(s-r_1)(s-r_2) = \phi$

ex/ $I_o(s) = \frac{5s+1}{s+1} \rightarrow$ if $m \geq N$ (if highest exponent of s in numerator ≥ same in denom)

Use polynomial long division first

$$I_o(s) = \frac{5s+1}{s+1}$$

$$= 5 + \frac{-4}{s+1}$$

$$s+1 \overline{\big)\,5s+1}\quad 5$$

$$-\;5s-5$$

$$-4 \quad \text{Remainder}$$

$$\mathcal{L}^{-1}\left\{5 + \frac{-4}{s+1}\right\} = \boxed{5\delta(t) - 4e^{-t}u(t) = i_o(t)}$$

Transfer Functions

factor

$$H(s) = \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i} = \frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}$$

polynomial form — factored pole/zero form

roots of numerator, z_i, are called zeros
- values of s at which $H(s) = \emptyset$

roots of denominator, p_i, are called poles
- values of s near which $H(s) \to \infty$

if all a_i are real, then all z_i are either real or complex conjugate pairs
same is true for b_i & p_i

Both true for models of real circuits

Poles define the terms in $H(s)$ → which become terms in $\mathcal{L}^{-1}\{V_I(s) \cdot H(s)\}$
zeros come into play in determining residues

Partial Fraction Expansion / Decomposition

$\underline{M < N}$ (otherwise do long division first)

$$\frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i} = \frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)} \overset{PFE}{=} \frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_2)} + \cdots \frac{k_N}{(s-p_N)}$$

polynomial form — factored pole-zero form — Partial Fraction Expansion

k_i are called "residues"

if all p_i are real & distinct & $N > M$

→ then find k_i by "coverup" method → Multiply both sides by $(s-p_i)$ then evaluate at $s = p_i$

ex for k_2

$$\left.\frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)\cancel{(s-p_2)}\cdots(s-p_N)}\cancel{(s-p_2)}\right|_{s=p_2} = \left.\frac{k_1\cancel{(s-p_2)}}{\cancel{(s-p_1)}}\right|_{s=p_2} + \frac{k_2\cancel{(s-p_2)}}{\cancel{(s-p_2)}} + \cdots \left.\frac{k_N\cancel{(s-p_2)}}{\cancel{(s-p_N)}}\right|_{s=p_2}$$

ex/ $$F(s) = \frac{4(s+2)}{s^2+4s+3} \overset{factor}{=} \frac{4(s+2)}{(s+1)(s+3)} \overset{PFE}{=} \frac{k_1}{s+1} + \frac{k_2}{s+3} = \frac{2}{s+1} + \frac{2}{s+3}$$

$$k_1 = \left.\frac{4(s+2)}{(s+3)}\right|_{s=-1} = 2 = k_1$$

$$k_2 = \left.\frac{4(s+2)}{(s+1)}\right|_{s=-3} = 2 = k_2$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2e^{-t}u(t) + 2e^{-3t}u(t)$$