

Laplace and Fourier Revisited

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} f(t) dt$$

Laplace Transform (Bilateral):

$$F(s) = \int_{-\infty}^{+\infty} e^{-st} f(t) dt \quad s = \sigma + j\omega$$

(Handwritten note: $s \rightarrow j\omega$ ($\sigma = 0$))

If we let $s \rightarrow j\omega$ the Laplace transform & Fourier transform are equivalent, but for some signals the integrals won't converge

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

Inverse Laplace Transform (Bilateral):

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st} ds$$

(Handwritten note: $s \rightarrow j\omega$)

Laplace Explanation

$$F(s) = \int_0^{-} e^{-st} f(t) dt = \int_0^{+\infty} e^{-\sigma t} e^{-j\omega t} f(t) dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{\sigma t} e^{j\omega t} F(s) ds$$

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Fourier Series

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(k\omega_0 t) + b_n \sin(k\omega_0 t)$$

1. $f(t)$ is single-valued
2. $f(t)$ is finite
3. $f(t)$ has finite discontinuities and maxima/min in any closed interval

Need to find a_0, a_n, b_n for some function $f(t)$

for a_0 : $a_0 = \frac{1}{T} \int_0^T f(t) dt$ a_0 is average / DC value of $f(t)$

For a_n : $a_n \rightarrow$ but not $\frac{1}{T} \int_0^T f(t) \cos(n\omega_0 t) dt$

plugging in Fourier Series for $f(t)$:

$$\frac{1}{T} \int_0^T f(t) \cos(n\omega_0 t) dt = \frac{1}{T} \int_0^T \left[a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right] \cos(n\omega_0 t) dt$$

$$= \frac{1}{T} \int_0^T a_0 \cos(n\omega_0 t) dt + \frac{1}{T} \int_0^T \left[\sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) \cos(n\omega_0 t) + b_k \sin(k\omega_0 t) \cos(n\omega_0 t) \right] dt$$

<https://www.kitandkid.com/fourier/fourier.html>

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Non-periodic Waveforms: Fourier Transform

Fourier Series \rightarrow work only for periodic waveforms
 Fourier Transform \rightarrow for non-periodic signals
 Idea: trial any non-periodic signal as if it was periodic with $T \rightarrow \infty$

Fourier Series: $C_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt$
 Fourier Transform: $T C_n = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

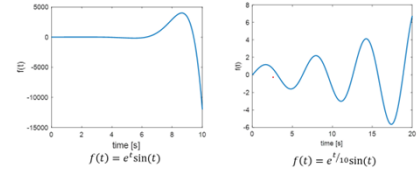
Fourier Series: summation $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$
 Smart Fourier Transform: $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$

1. $f(t)$ can be expressed this way if
2. $f(t)$ is single-valued
3. $f(t)$ is finite
4. $f(t)$ has finite discontinuities and maxima/min in any closed interval

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Unbounded Signals & Unstable Systems



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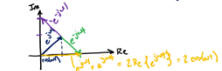
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Complex Form of Fourier Series

Euler: $e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$



Plug into Fourier Series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + \frac{b_n}{2j} (e^{jn\omega_0 t} - e^{-jn\omega_0 t})$$

$$= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n}{2j} \right) e^{jn\omega_0 t} + \left(\frac{a_n}{2} + \frac{b_n}{2j} \right) e^{-jn\omega_0 t}$$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \rightarrow C_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt \quad C_n^* = C_{-n}$$

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Example Signal Laplace Transforms

$f(t) = u(t)$ $\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} u(t) dt = \int_0^{\infty} e^{-st} dt$
 $= \left[-\frac{1}{s} e^{-st} \right]_0^{\infty} = \left[0 - \left(-\frac{1}{s}\right) \right]$
 $F(s) = \mathcal{L}\{u(t)\} = \frac{1}{s} \quad \text{if } \operatorname{Re}\{s\} > 0$
 Region of convergence for $\mathcal{L}\{u(t)\} \rightarrow \operatorname{Re}\{s\} > 0$
 $s = \sigma + j\omega \rightarrow \sigma > 0$

$f(t) = e^{-at} u(t)$ $\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} e^{-at} u(t) dt = \int_0^{\infty} e^{-(s+a)t} dt$
 $\mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a} \quad \text{if } \operatorname{Re}\{s+a\} > 0$

Generalize: $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$
 (where $F(s) = \mathcal{L}\{f(t)\}$)

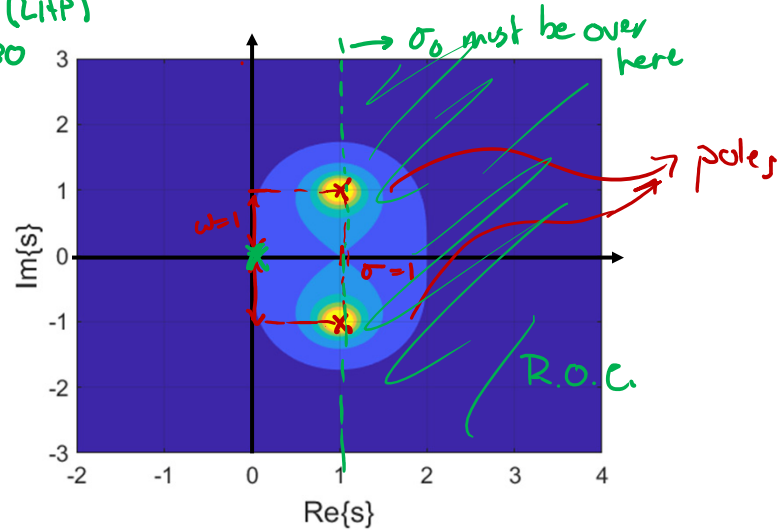
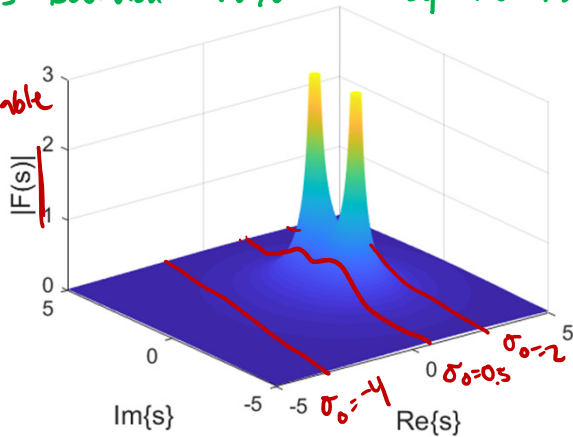
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The s-plane

The Region of convergence of a Laplace transform is the complex plane to the right of all poles

If all poles are in the open left half plane, (LHP) the signal is bounded and/or the system is BIBO stable

RHP poles \Leftrightarrow unstable



$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s - (1 + j))(s - (1 - j))}$$

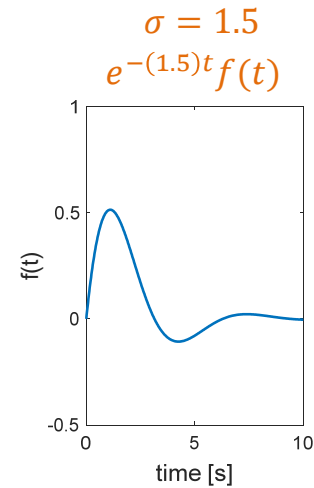
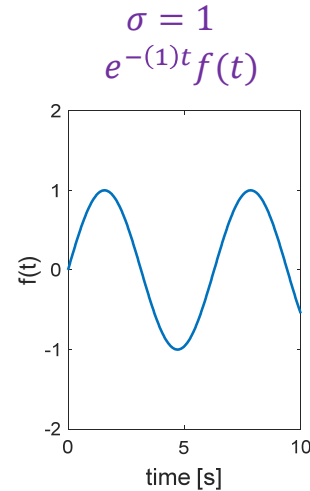
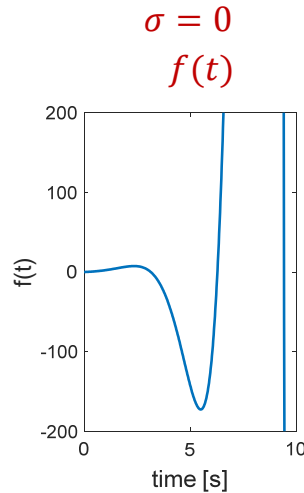
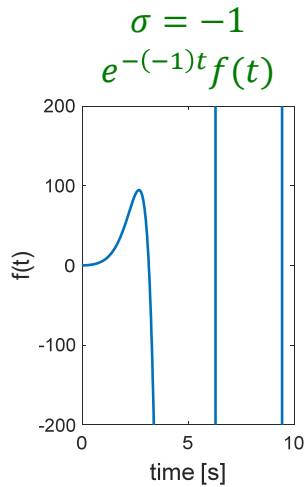
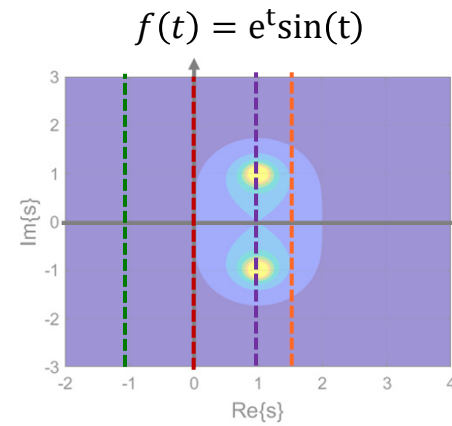
$$f(t) = e^t \sin(t)$$

$$F(s) = \int_0^{\infty} e^{-st} e^t \sin(t) dt \rightarrow \text{Need } \text{Re}\{s\} > 1 \text{ for convergence}$$

ROC is $\text{Re}\{s\} > 1$

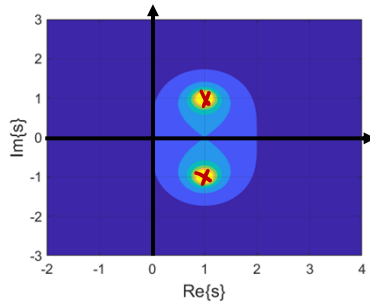
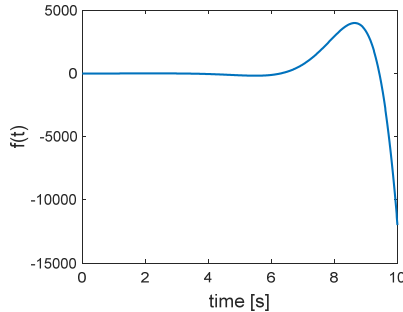
Example R.O.C.

$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t) dt = \int_{0^-}^{+\infty} e^{-j\omega t} e^{-\sigma t} f(t) dt$$



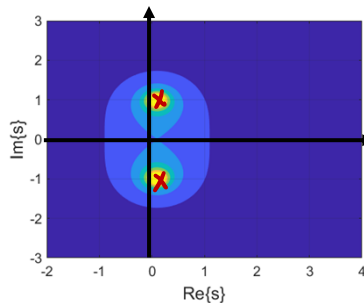
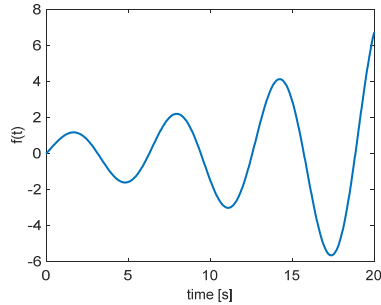
Example Functions

$$f(t) = e^t \sin(t)$$



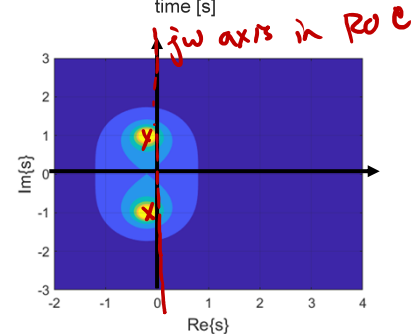
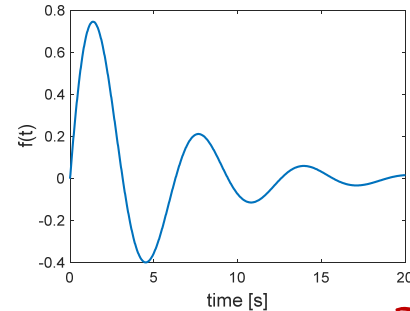
$$F(s) = \frac{1}{(s - (1 + j))(s - (1 - j))}$$

$$f(t) = e^{t/10} \sin(t)$$



$$F(s) = \frac{1}{\left(s - \left(\frac{1}{10} + j\right)\right)\left(s - \left(\frac{1}{10} - j\right)\right)}$$

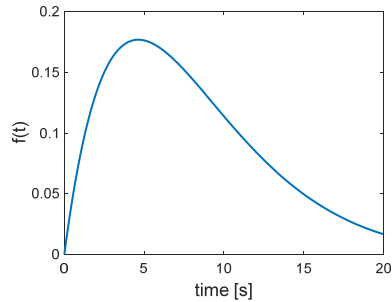
$$f(t) = e^{-t/5} \sin(t)$$



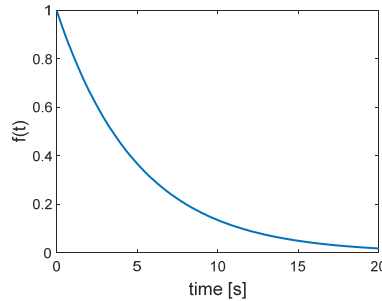
$$F(s) = \frac{1}{\left(s + \left(\frac{1}{5} + j\right)\right)\left(s + \left(\frac{1}{5} - j\right)\right)}$$

Example Functions

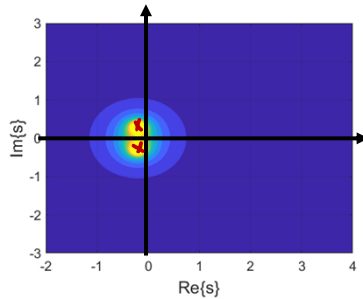
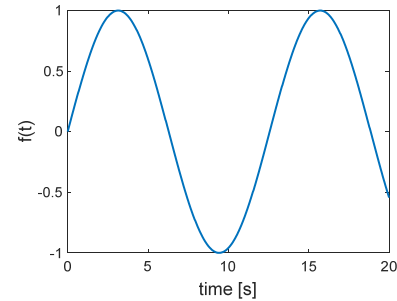
$$f(t) = e^{-t/5} \sin(t/10)$$



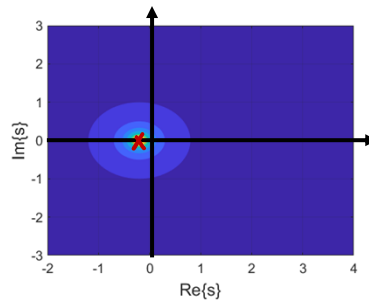
$$f(t) = e^{-t/5}$$



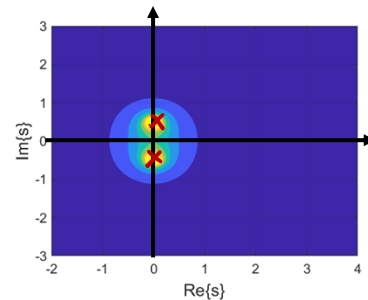
$$f(t) = \sin(t/2)$$



$$F(s) = \frac{1/10}{\left(s + \left(\frac{1}{5} + \frac{j}{10}\right)\right)\left(s + \left(\frac{1}{5} - \frac{j}{10}\right)\right)}$$



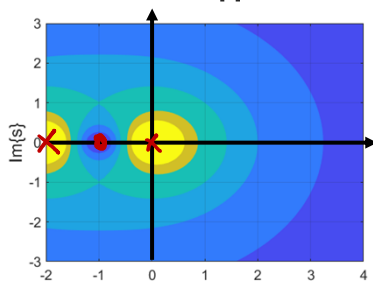
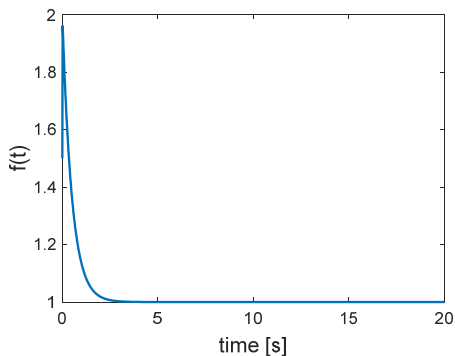
$$F(s) = \frac{1}{\left(s + \frac{1}{5}\right)}$$



$$F(s) = \frac{1/2}{\left(s + \frac{j}{2}\right)\left(s - \frac{j}{2}\right)}$$

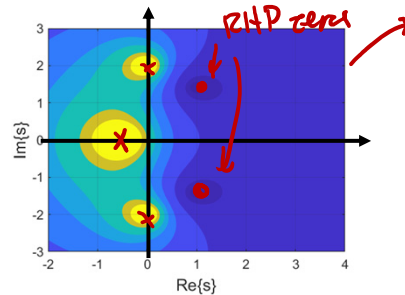
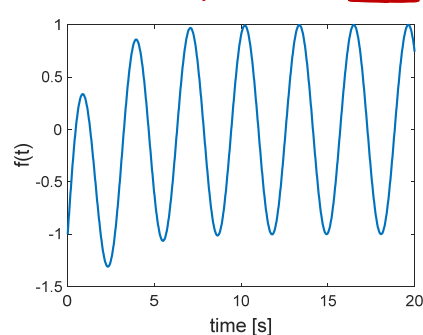
Example Functions

$$f(t) = -e^{-t/2} + u(t)$$



$$F(s) = 2 \frac{s+1}{s(s+2)}$$

$$f(t) = -e^{-t/2} + \sin(2t)$$

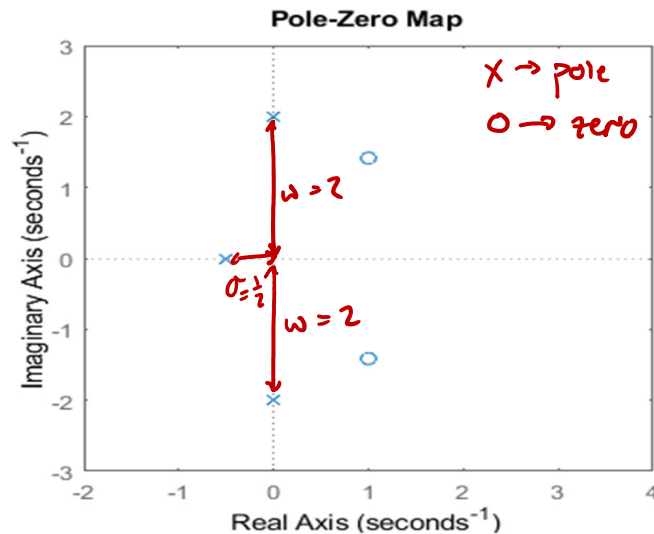
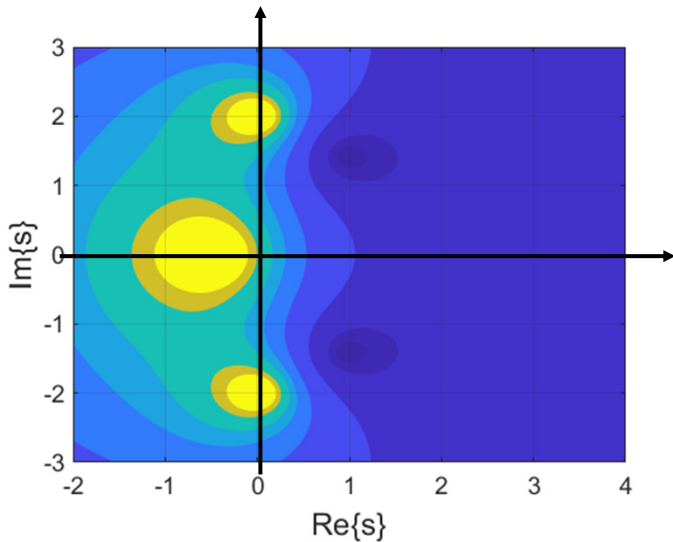


$$F(s) = -\frac{(s+(-1+j\sqrt{2}))(s+(-1-j\sqrt{2}))}{(s+1/2)(s+j2)(s-j2)}$$

RHP zeros do not
make a signal diverge
May also see these called
Non-minimum phase zeros

Pole-Zero Map

$$F(s) = -\frac{(s + (-1 + j\sqrt{2}))(s + (-1 - j\sqrt{2}))}{(s + 1/2)(s + j2)(s - j2)}$$



MATLAB:

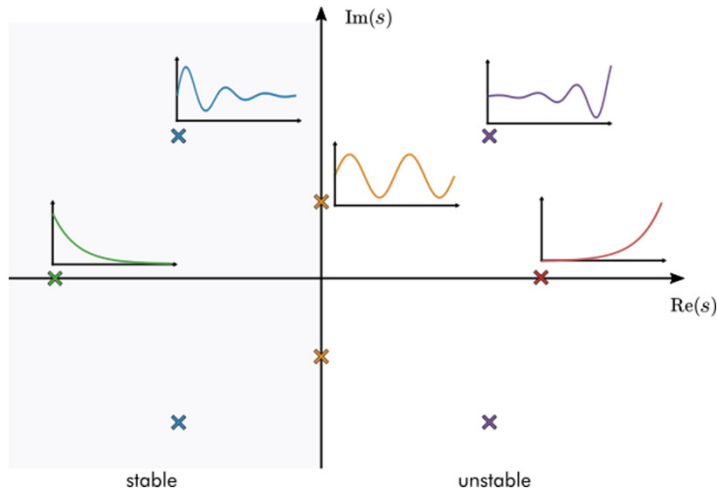
```
[R,X] = meshgrid(-2:.01:4,-3:.01:3);  
s = R + 1j*X;  
Fs = (s.^2-2*s+3)./(s.^2 + 4)./(s + 1/2);  
[C,h] = contourf(R,X,abs(Fs));
```

```
h.LevelList = [0 .05 .1 .25 .5 1 1.5 2];  
h.LineStyle = 'none';
```

MATLAB:

```
s = tf('s');  
Fs = 2/(s^2 + 4) - 1/(s + 1/2);  
pzmap(Fs);
```


Poles-Zero Plot



Takeaways:

1. Pole location tells us the "form" of our function
2. Complex poles/zeros & their residues always show up as conjugate pairs (for real signals/systems)

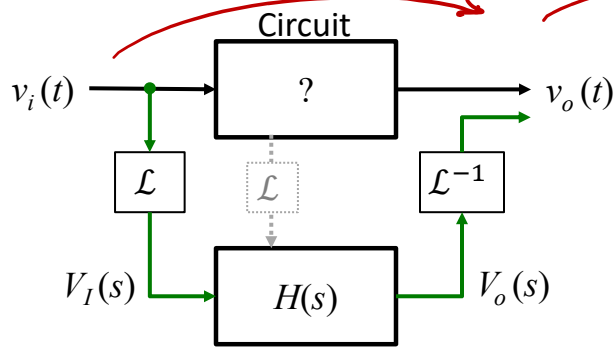
3. If all poles are in the open LHP
 - signal is bounded
 - system/circuit is BIBO stable

If any pole is in RHP \rightarrow ^{unbounded} unstable

If poles on $j\omega$ -axis, need to look at multiplicity

4. If all poles in open LHP, $j\omega$ -axis is in region of convergence $H(s \rightarrow j\omega)$ is Freq. resp.

System I/O Relationship



201 approach \rightarrow solve Diff Eqs

$$\mathcal{L}\{v_i(t)\} = V_I(s)$$

Take the Laplace transform of the circuit
 \downarrow solve it to get $H(s)$

$$V_O(s) = H(s) V_I(s)$$

$$v_o(t) = \mathcal{L}^{-1}\{V_O(s)\}$$

$$v_o(t) = \mathcal{L}^{-1}\{V_I(s) H(s)\}$$

What is $\mathcal{L}^{-1}\{H(s)\}$?

Look at what happens if
 the $V_O(s) = H(s) \cdot 1$ \neq

$h(t) \rightarrow$ impulse response of circuit

$$v_i(t) = \delta(t) \rightarrow \mathcal{L}\{\delta(t)\} = 1$$

$$v_o(t) = h(t)$$