Laplace and Fourier Revisited

Fourier Transform: $$F(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} f(t)\, dt$$

$s \to j\omega$ $(\sigma = 0)$

Laplace Transform (Bilateral): $$F(s) = \int_{-\infty}^{+\infty} e^{-st} f(t)\, dt$$

$s = \sigma + j\omega$

If we let $s \to j\omega$ the laplace transform & fourier transform are equivalent, but for some signals the integrals won't converge

Inverse Fourier Transform: $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

$s \to j\omega$

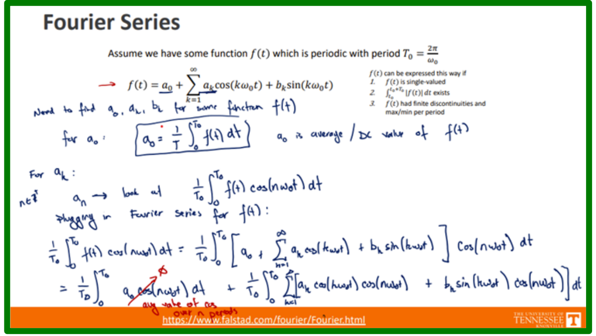
Inverse Laplace Transform (Bilateral): $$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st} ds$$

Laplace Explanation

$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t)\, dt = \int_{0^-}^{+\infty} e^{-\sigma t} e^{-j\omega t} f(t) dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{\sigma t} e^{j\omega t} F(s)\, ds$$

L21



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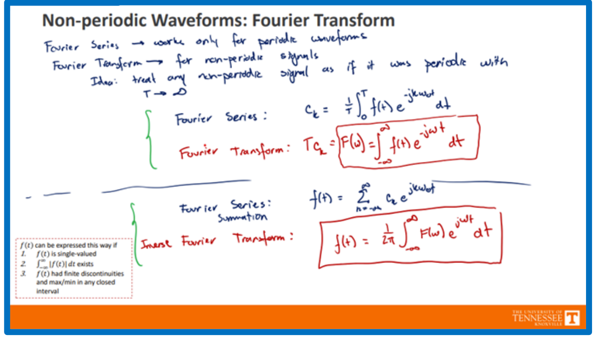

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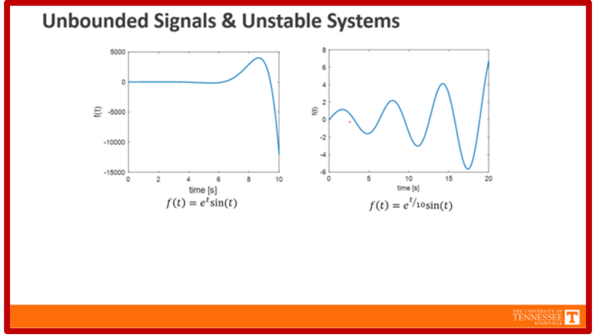

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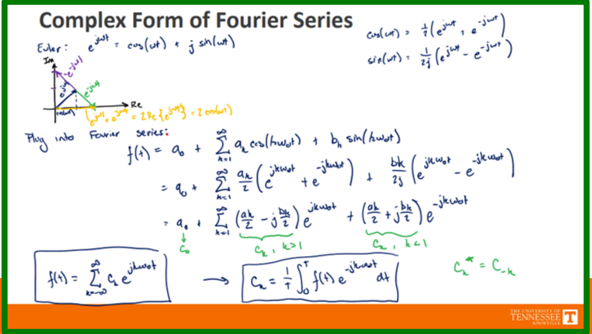

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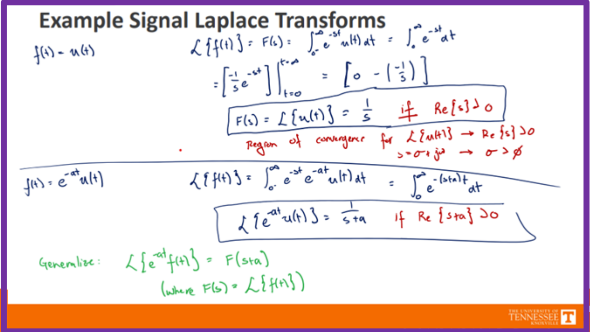
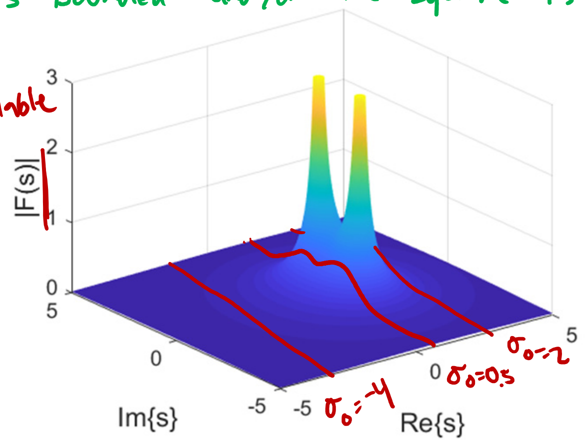


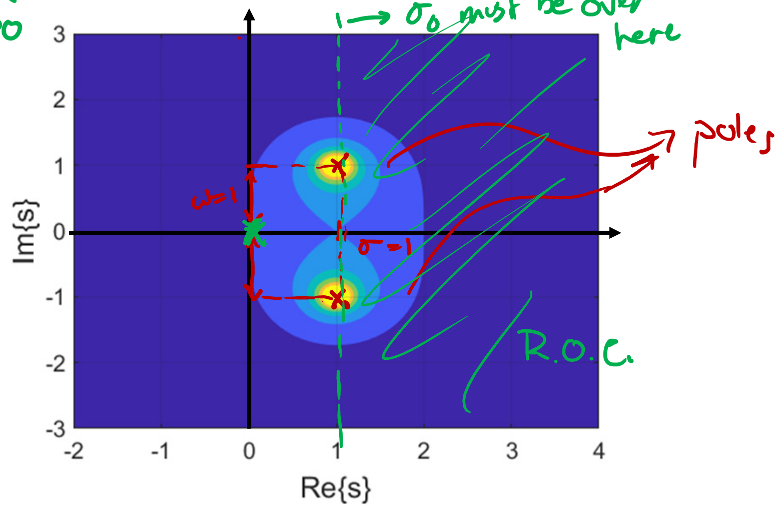
The s-plane

The Region of convergence of a Laplace transform is the complex plane to the right of all poles

If all poles are in the open left half-plane, (LHP) the signal is bounded and/or the system is BIBO stable

RHP poles $\Leftrightarrow$ Unstable





$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s-(1+j))(s-(1-j))}$$

ROC is $Re\{s\} > 1$

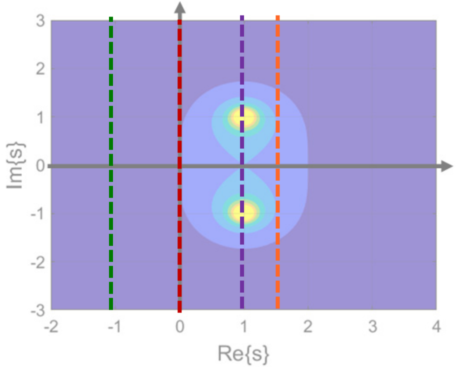
$f(t) = e^t \sin(t)$

$F(s) = \int_{0^-}^{\infty} \underbrace{e^{-st}\, e^{t}} \sin(t)\, dt \rightarrow$ Need $Re\{s\} > 1$ for convergence

Example R.O.C.

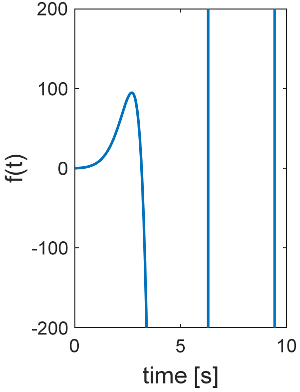
$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t)\, dt = \int_{0^-}^{+\infty} e^{-j\omega t} e^{-\sigma t} f(t) dt$$

$f(t) = e^t \sin(t)$



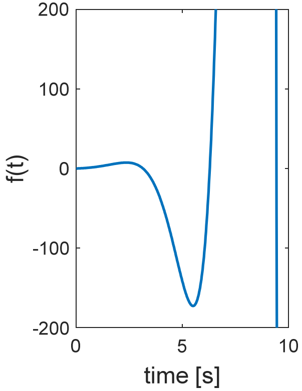
$\sigma = -1$

$e^{-(-1)t} f(t)$



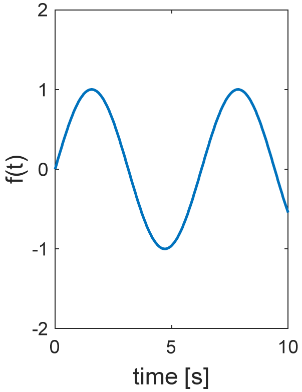
$\sigma = 0$

$f(t)$



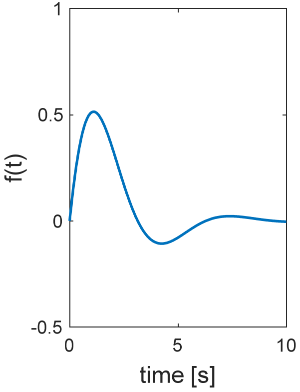
$\sigma = 1$

$e^{-(1)t} f(t)$



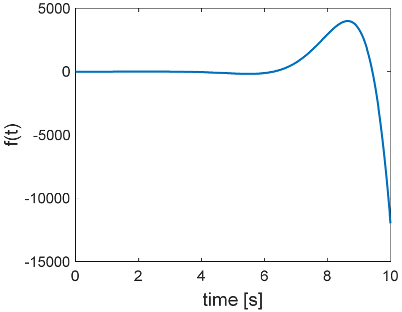
$\sigma = 1.5$

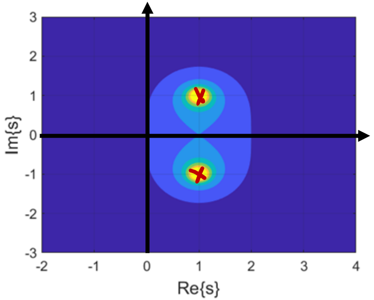
$e^{-(1.5)t} f(t)$



Example Functions

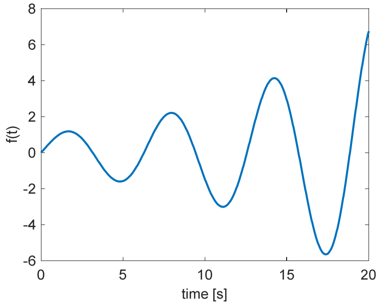
$$f(t) = e^{t}\sin(t)$$

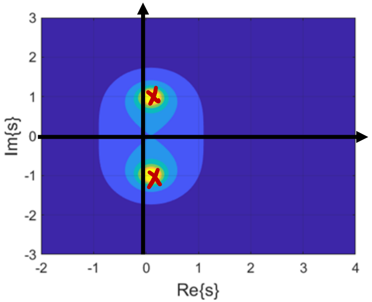




$$F(s) = \frac{1}{(s-(1+j))(s-(1-j))}$$

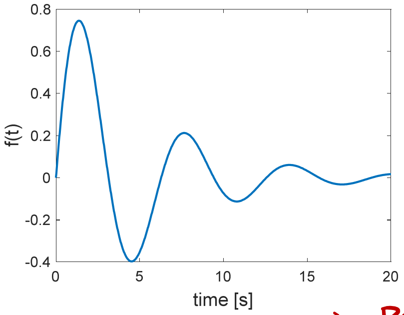
$$f(t) = e^{t/10}\sin(t)$$

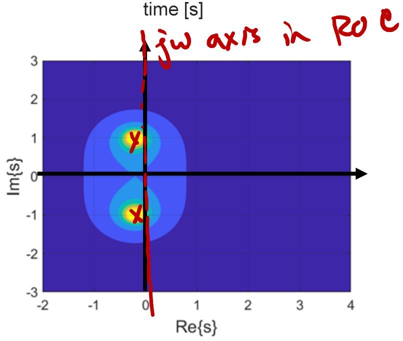




$$F(s) = \frac{1}{\left(s-\left(\frac{1}{10}+j\right)\right)\left(s-\left(\frac{1}{10}-j\right)\right)}$$

$$f(t) = e^{-t/5}\sin(t)$$

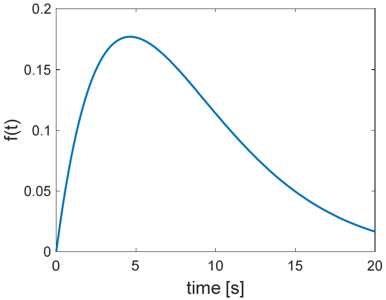


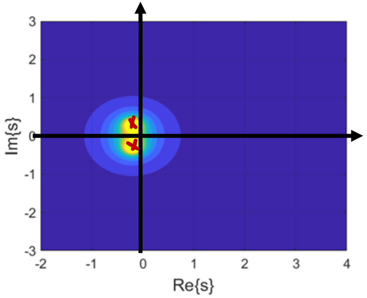


$$F(s) = \frac{1}{\left(s+\left(\frac{1}{5}+j\right)\right)\left(s+\left(\frac{1}{5}-j\right)\right)}$$

Example Functions

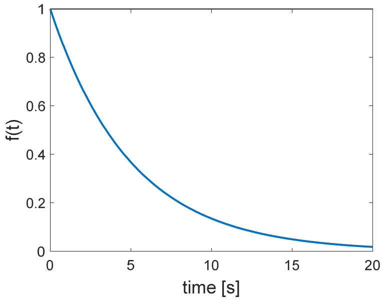
$$f(t) = e^{-t/5}\sin\left(t/_{10}\right)$$

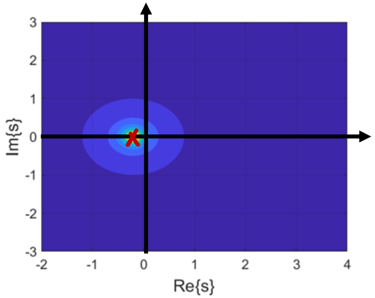




$$F(s) = \frac{1/_{10}}{\left(s + \left(\frac{1}{5} + \frac{j}{10}\right)\right)\left(s + \left(\frac{1}{5} - \frac{j}{10}\right)\right)}$$

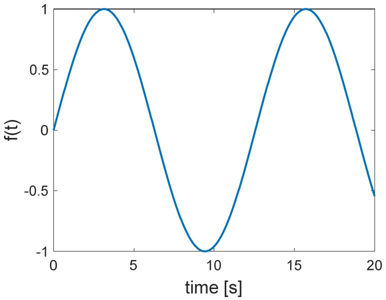
$$f(t) = e^{-t/5}$$

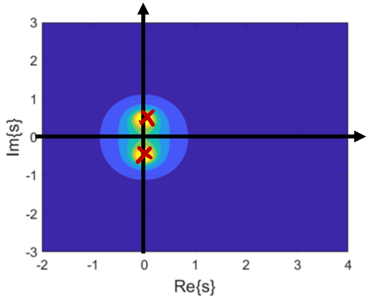




$$F(s) = \frac{1}{\left(s + \frac{1}{5}\right)}$$

$$f(t) = \sin\left(t/_{2}\right)$$

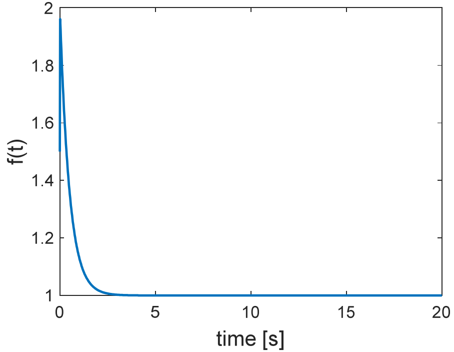


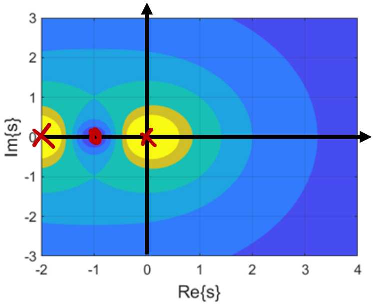


$$F(s) = \frac{1/_{2}}{\left(s + \frac{j}{2}\right)\left(s - \frac{j}{2}\right)}$$

Example Functions

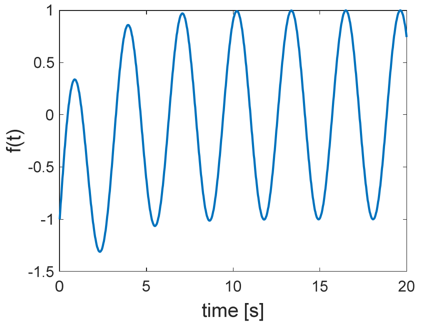
$$f(t) = -e^{-t/2} + u(t)$$

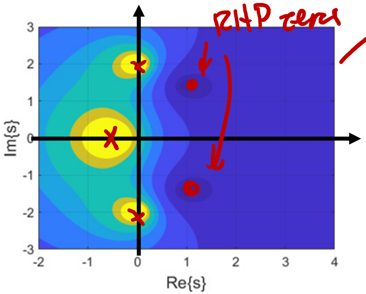




$$F(s) = 2\frac{s+1}{s(s+2)}$$

$$f(t) = -e^{-t/2} + \sin(2t)$$





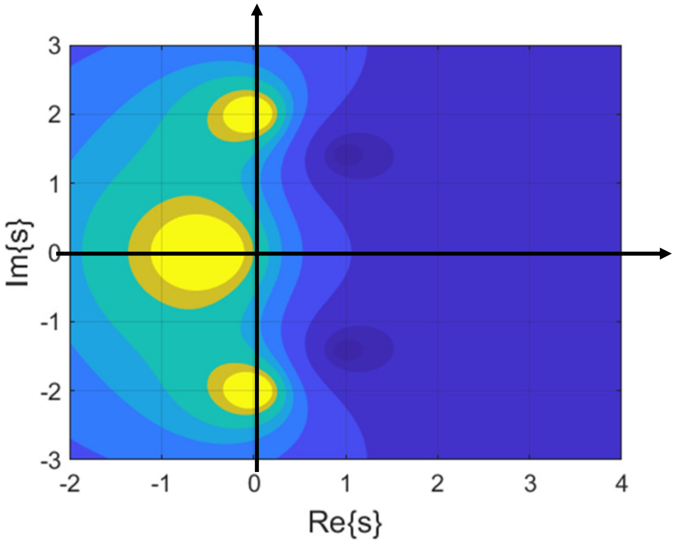
$$F(s) = -\frac{\left(s + \left(-1 + j\sqrt{2}\right)\right)\left(s + \left(-1 - j\sqrt{2}\right)\right)}{\left(s + {}^{1}/_{2}\right)(s + j2)(s - j2)}$$

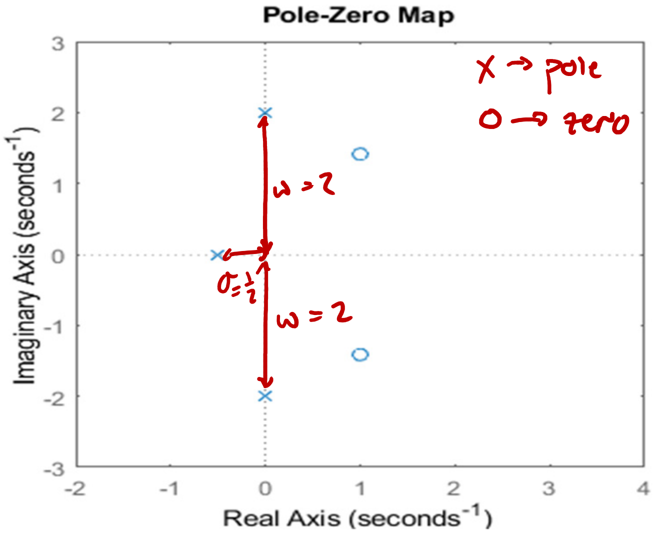
RHP zeros do not make a signal diverge

May also see these called Non-minimum phase zeros

Pole-Zero Map

$$F(s) = -\frac{\left(s + \left(-1 + j\sqrt{2}\right)\right)\left(s + \left(-1 - j\sqrt{2}\right)\right)}{\left(s + {}^{1}/_{2}\right)(s + j2)(s - j2)}$$





MATLAB:

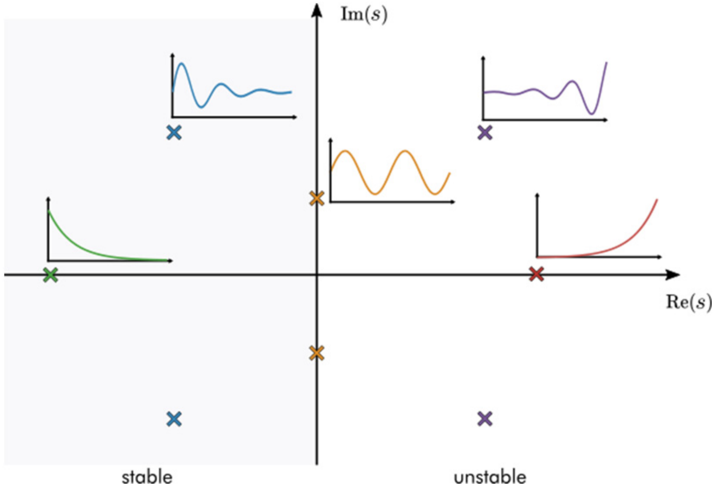
```
[R,X] = meshgrid(-2:.01:4,-3:.01:3);
s = R + 1j*X;
Fs = (s.^2-2*s+3)./(s.^2 + 4)./(s + 1/2);
[C,h] = contourf(R,X,abs(Fs));


h.LevelList = [0 .05 .1 .25 .5 1 1.5 2];
h.LineStyle = 'none';
```

MATLAB:

```
s = tf('s');
Fs = 2/(s^2 + 4) - 1/(s + 1/2);
pzmap(Fs);
```

Poles-Zero Plot



Takeaways:

1. Pole location tells us the "form" of our function

2. Complex poles / zeros & their residues always show up as conjugate pairs (for real signals / systems)

3. If all poles are in the open LHP
   - signal is bounded
   - system / circuit is BIBO stable

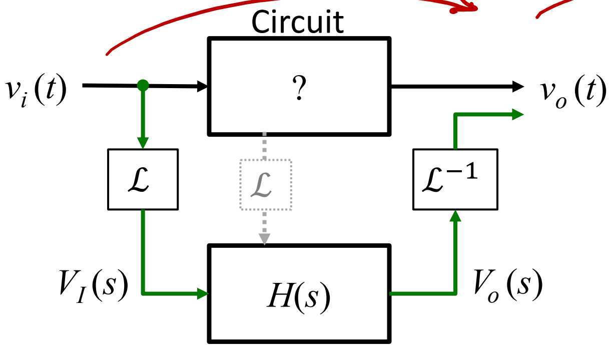
   If any pole is in RHP → unbounded / unstable

   If pole(s) on $j\omega$-axis, need to look at multiplicity

4. If all poles in open LHP, $j\omega$-axis is in region of convergence, $H(s \to j\omega)$ is Freq. Resp.

System I/O Relationship

201 approach → solve Diff EQs



$\mathcal{L}\{v_i(t)\} = V_I(s)$

Take the Laplace transform of the circuit & solve it to get $H(s)$

$V_o(s) = H(s)\,V_I(s)$

$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\}$

$v_o(t) = \mathcal{L}^{-1}\{V_I(s)H(s)\}$

what is $\mathcal{L}^{-1}\{H(s)\}$?

$\boxed{h(t) \longrightarrow \text{impulse response of circuit}}$

Look at what happens if $v_i(t) = \delta(t)$? → $\mathcal{L}\{\delta(t)\} = 1$

the $V_o(s) = H(s)\,(1)$ & $v_o(t) = h(t)$