

# Laplace and Fourier Revisited

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} f(t) dt$$

Laplace Transform (Bilateral):

$$F(s) = \int_{-\infty}^{+\infty} e^{-st} f(t) dt$$

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Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

Inverse Laplace Transform (Bilateral):

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} F(s) e^{st} ds$$

# Laplace Explanation

$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t) dt$$

$$\int_{0^-}^{+\infty} e^{-\sigma t} e^{-j\omega t} f(t) dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} e^{\sigma t} e^{j\omega t} F(s) ds$$

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## Fourier Series

Assume we have some function  $f(t)$  which is periodic with period  $T_0 = \frac{2\pi}{\omega_0}$

$\rightarrow f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(k\omega_0 t) + b_n \sin(k\omega_0 t)$

Want to find  $a_0, a_n, b_n$  for some function  $f(t)$

for  $a_0$ :  $a_0 = \frac{1}{T_0} \int_{0}^{T_0} f(t) dt$   $a_0$  is average / DC value of  $f(t)$

For  $a_n$ :

$a_n \rightarrow$  look at  $\frac{1}{T_0} \int_{0}^{T_0} f(t) \cos(n\omega_0 t) dt$   
 Plugging in Fourier Series for  $f(t)$ :

$$\begin{aligned} \frac{1}{T_0} \int_{0}^{T_0} f(t) \cos(n\omega_0 t) dt &= \frac{1}{T_0} \int_{0}^{T_0} \left[ a_0 + \sum_{n=1}^{\infty} a_n \cos(k\omega_0 t) + b_n \sin(k\omega_0 t) \right] \cos(n\omega_0 t) dt \\ &= \frac{1}{T_0} \int_{0}^{T_0} a_0 \cos(n\omega_0 t) dt + \frac{1}{T_0} \int_{0}^{T_0} \left[ \sum_{n=1}^{\infty} a_n \cos(k\omega_0 t) \cos(n\omega_0 t) + b_n \sin(k\omega_0 t) \cos(n\omega_0 t) \right] dt \end{aligned}$$

<https://www.falstad.com/fourier/Fourier.html>

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## Non-periodic Waveforms: Fourier Transform

Fourier Series → works only for periodic waveforms

Fourier Transform → for non-periodic signals  
 Idea: treat any non-periodic signal as if it was periodic with  $T \rightarrow \infty$

Fourier Series:  $c_k = \frac{1}{T_0} \int_0^{T_0} f(t) e^{-jk\omega_0 t} dt$

Fourier Transform:  $T_0 = \int_{-\infty}^{\infty} |F(\omega)|^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

Fourier Series Summation:

Smart Fourier Transform:

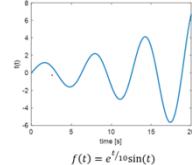
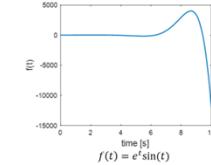
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$f(t)$  can be expressed this way if:  
 1.  $f(t)$  is single-valued  
 2.  $\int_{-\infty}^{\infty} |f(t)|^2 dt$  exists  
 3.  $f(t)$  has finite discontinuities and min in any closed interval

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## Unbounded Signals & Unstable Systems



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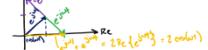
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## Complex Form of Fourier Series

$$\text{Euler: } e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

$$\cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})$$



Plug into Fourier series:

$$\begin{aligned} f(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(k\omega_0 t) + b_n \sin(k\omega_0 t) \\ &= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} \left( e^{jk\omega_0 t} + e^{-jk\omega_0 t} \right) + \frac{b_n}{2j} \left( e^{jk\omega_0 t} - e^{-jk\omega_0 t} \right) \\ &= a_0 + \sum_{n=1}^{\infty} \left( \frac{a_n}{2} + \frac{jb_n}{2} \right) e^{jk\omega_0 t} + \left( \frac{a_n}{2} - \frac{jb_n}{2} \right) e^{-jk\omega_0 t} \end{aligned}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$\rightarrow c_n = \frac{1}{T_0} \int_0^{T_0} f(t) e^{-jn\omega_0 t} dt$$

$$c_n^* = c_{-n}$$

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## Example Signal Laplace Transforms

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt = \left[ \frac{1}{s} e^{-st} \right]_{t=0}^{t=\infty} = [0 - (-\frac{1}{s})]$$

$$F(s) = \mathcal{L}\{u(t)\} = \frac{1}{s} \quad \text{if } \text{Re}\{s\} > 0$$

Region of convergence for  $\mathcal{L}\{u(t)\} \rightarrow \text{Re}\{s\} > 0$   
 $\Rightarrow \sigma > 0 \Rightarrow \sigma > \rho$

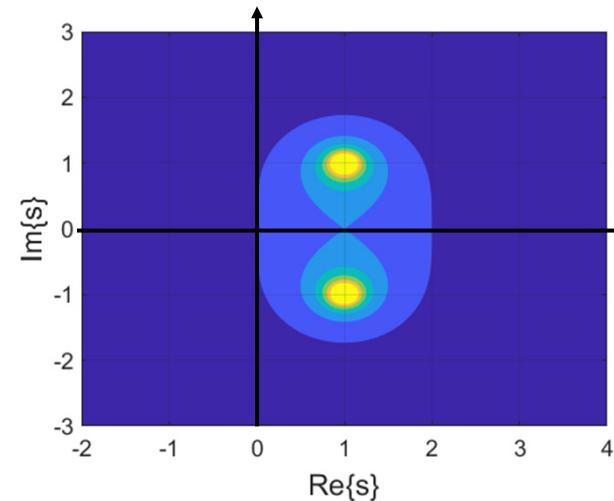
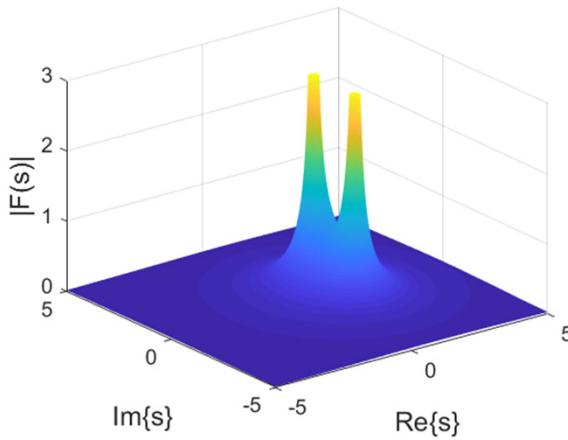
$$\mathcal{L}\{e^{-at} u(t)\} = \int_0^{\infty} e^{-st} e^{-at} u(t) dt = \int_0^{\infty} e^{-(s+a)t} dt$$

$$\mathcal{L}\{e^{at} u(t)\} = \frac{1}{s-a} \quad \text{if } \text{Re}\{s\} > a$$

Generalize:  $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$   
 (where  $F(s) = \mathcal{L}\{f(t)\}$ )

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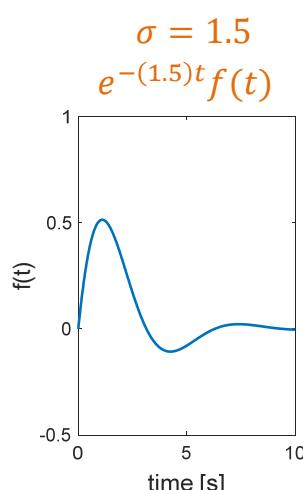
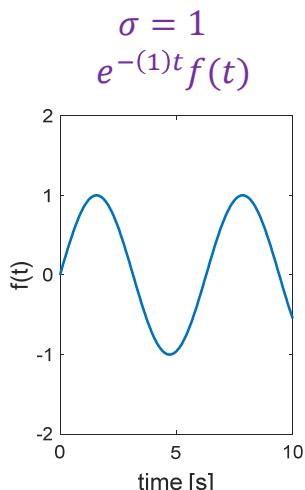
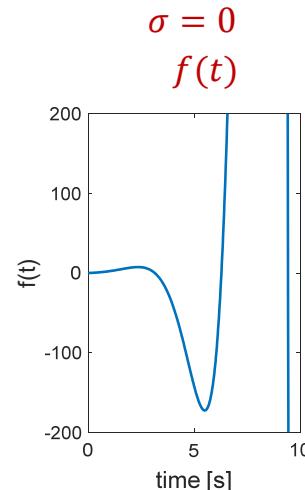
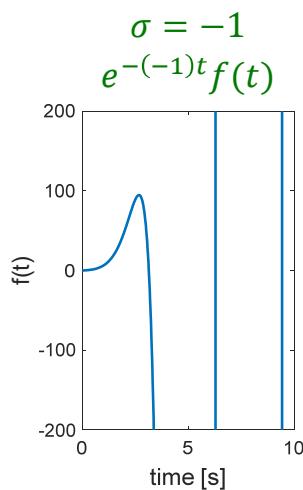
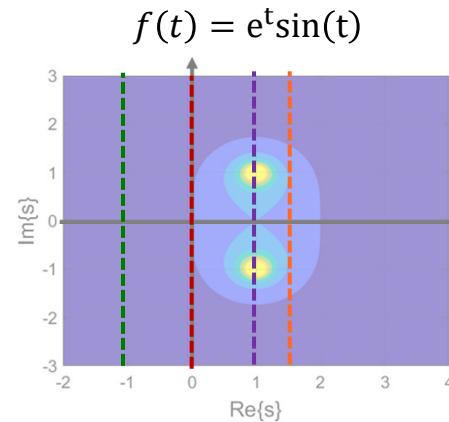
# The s-plane



$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s - (1 + j))(s - (1 - j))}$$

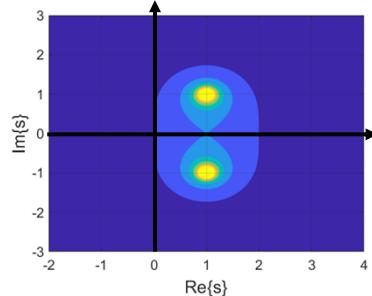
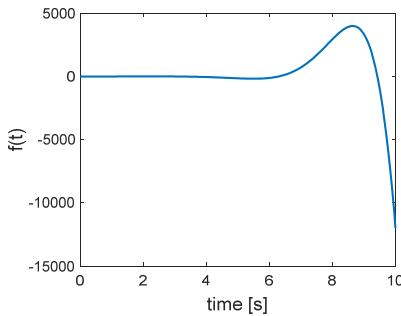
# Example R.O.C.

$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t) dt = \int_{0^-}^{+\infty} e^{-j\omega t} e^{-\sigma t} f(t) dt$$



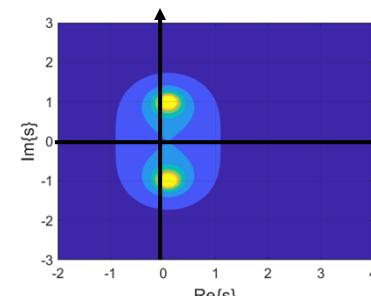
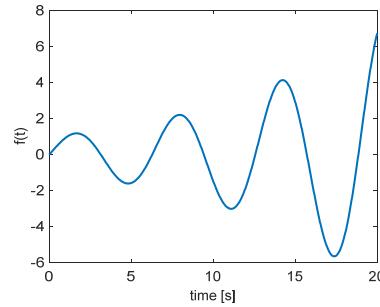
# Example Functions

$$f(t) = e^t \sin(t)$$



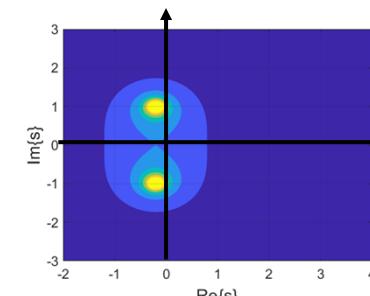
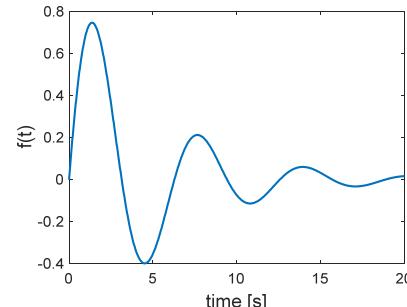
$$F(s) = \frac{1}{(s - (1 + j))(s - (1 - j))}$$

$$f(t) = e^{t/10} \sin(t)$$



$$F(s) = \frac{1}{\left(s - \left(\frac{1}{10} + j\right)\right)\left(s - \left(\frac{1}{10} - j\right)\right)}$$

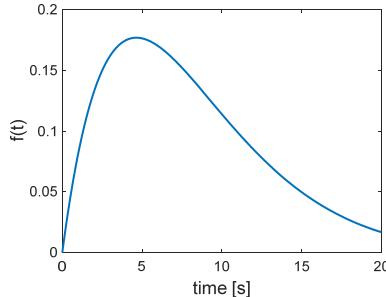
$$f(t) = e^{-t/5} \sin(t)$$



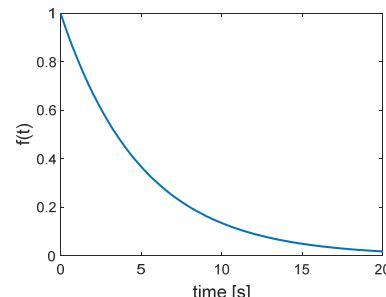
$$F(s) = \frac{1}{\left(s + \left(\frac{1}{5} + j\right)\right)\left(s + \left(\frac{1}{5} - j\right)\right)}$$

# Example Functions

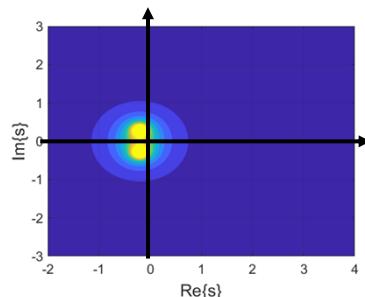
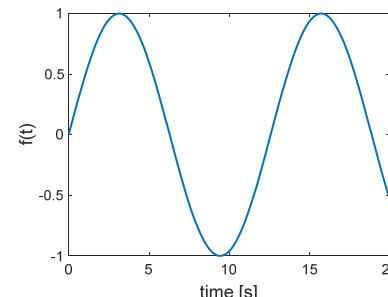
$$f(t) = e^{-t/5} \sin(t/10)$$



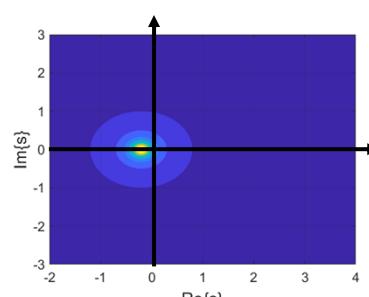
$$f(t) = e^{-t/5}$$



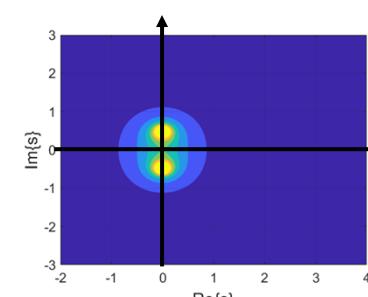
$$f(t) = \sin(t/2)$$



$$F(s) = \frac{1/10}{\left(s + \left(\frac{1}{5} + \frac{j}{10}\right)\right)\left(s + \left(\frac{1}{5} - \frac{j}{10}\right)\right)}$$

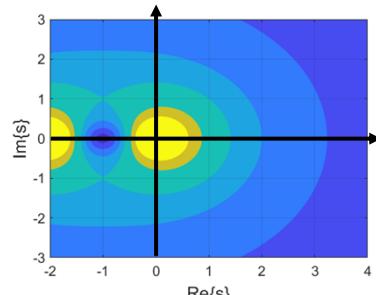
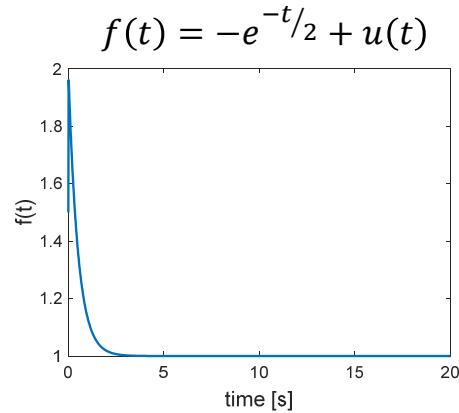


$$F(s) = \frac{1}{\left(s + \frac{1}{5}\right)}$$

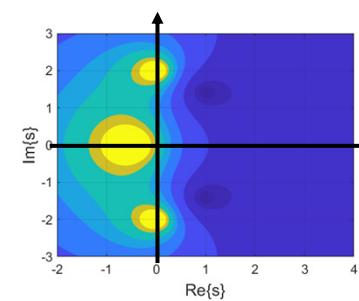
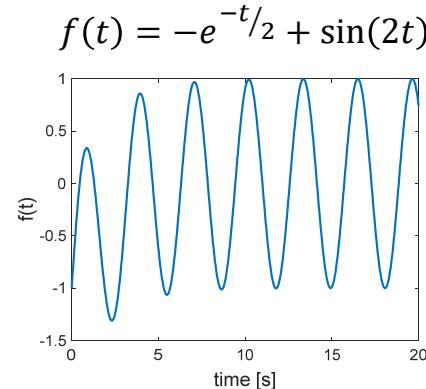


$$F(s) = \frac{1/2}{\left(s + \frac{j}{2}\right)\left(s - \frac{j}{2}\right)}$$

# Example Functions



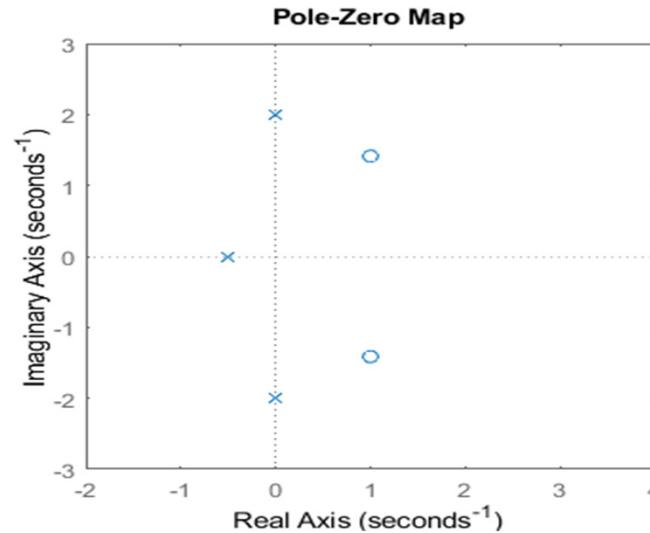
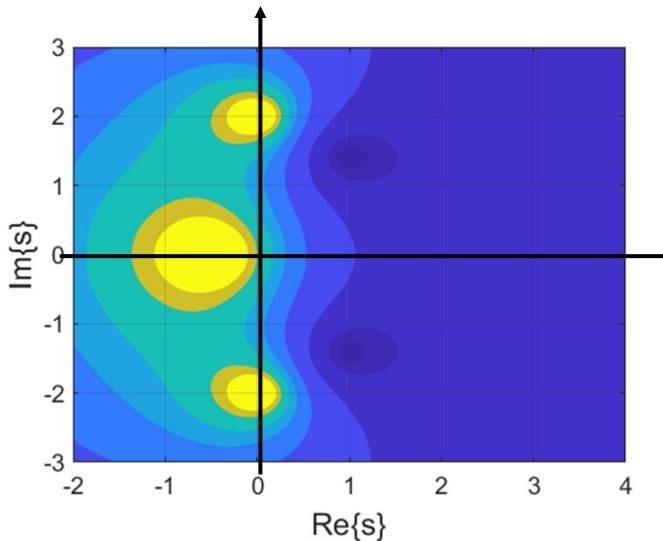
$$F(s) = 2 \frac{s+1}{s(s+2)}$$



$$F(s) = -\frac{(s + (-1 + j\sqrt{2})) (s + (-1 - j\sqrt{2}))}{(s + 1/2)(s + j2)(s - j2)}$$

# Pole-Zero Map

$$F(s) = -\frac{(s + (-1 + j\sqrt{2})) (s + (-1 - j\sqrt{2}))}{(s + 1/2)(s + j2)(s - j2)}$$



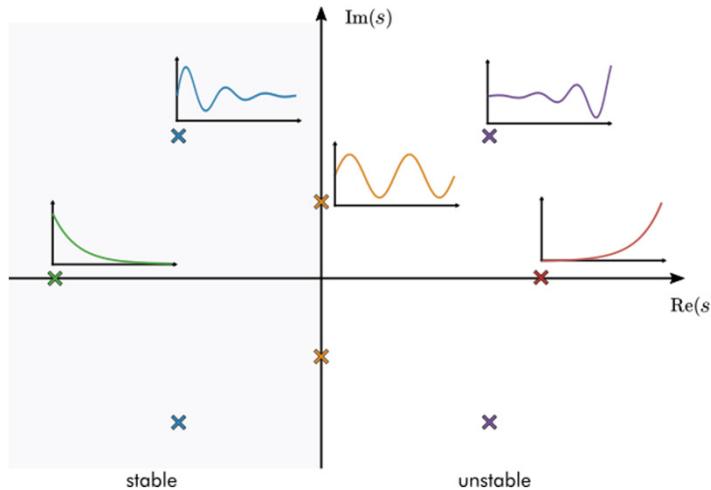
MATLAB:

```
s = tf('s');
Fs = 2/(s^2 + 4) - 1/(s + 1/2);
pzmap(Fs);
```

MATLAB:

```
s = tf('s');
Fs = 2/(s^2 + 4) - 1/(s + 1/2);
pzmap(Fs);
```

# Poles-Zero Plot



# System I/O Relationship

