Circuits II

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ECE 202 Lecture 36
April 24, 2024

THE UNIVERSITY OF
TENNESSEE
KNOXVILLE

Announcements

- Quiz 4
  - Added 5 points due to incorrect subscript in part (c)
- Experiment 4: Frequency Response
  - **Individual**, optional, extra credit
  - Posted after midterm – requires Section 15.9
  - No report, just turn in MATLAB and LTSpice files and a screenshot
- TNvoice Open
  - Please fill out
  - +5 pts EC on final for 100% response rate

Material for Midterm 2

- Midterm Exam #2 Friday Apr 26th
  - Lectures 20 - 33
  - Homeworks 6 - 9
  - Quiz 3 - 4
  - Chapter 17 (17.1 -17.5) and Chapter 14 (all)
  - Experiment 3
- Problems
  - Inverse Laplace transforms of a few *F*(*s*)
  - Inverse Laplace given input and system
  - Solve circuit transfer function including ICs Initial Conditions
  - For some systems and input signals, decide if the output is bounded

$$F(s) = \frac{175}{s^2}\left(\frac{50}{s^2+2s+50}\right)$$

$$p_{1,2} = \frac{-2 \pm \sqrt{4 - 4\,50}}{2} = -1 \pm j7$$

$$p_1 = -1 + j7$$

$$= \frac{175}{s^2}\,\frac{50}{(s^2+2s+50)} = \frac{175}{s^2}\left(\frac{50}{(s-(-1+j7))(s-(-1-j7))}\right)$$

$$= \frac{k_1}{s} + \frac{k_2}{s^2} + \frac{k_3}{s-p_1} + \frac{k_3^*}{s-p_1^*}$$

$$k_2 = \frac{175 \cdot 50}{s^2+2s+50}\bigg|_{s=0} = 175$$

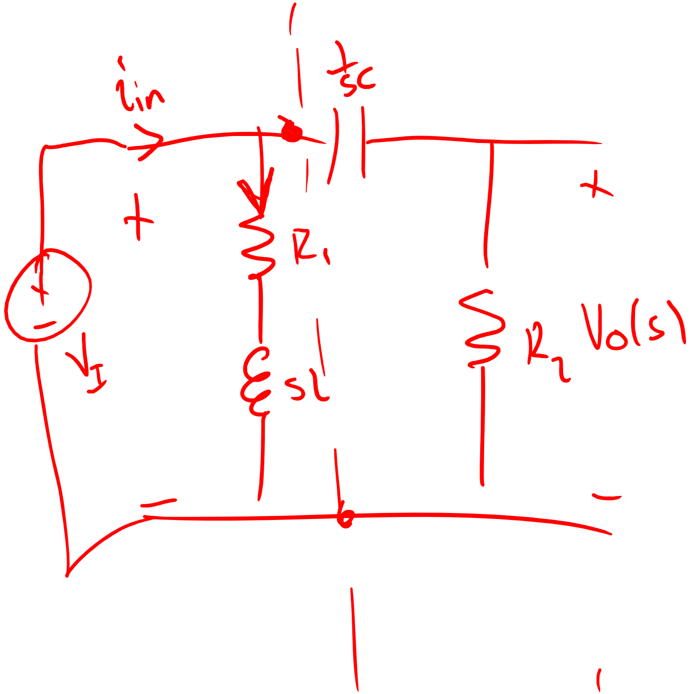
$$k_3 = \frac{175\cdot 50}{s^2\cdot(s-(-1-j7))}\bigg|_{s=-1+j7} = \frac{175\cdot 50}{(-1+j7)^2\cdot(j14)} = +3.5 + j12$$

$$|k| = 12.5$$

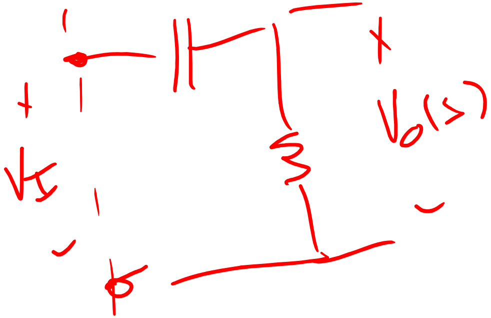
$$\angle k = 23^\circ$$

$$k_1 = \frac{d}{ds}\left[\frac{175\cdot 50}{-(s^2+2s+50)^2}(2s+2)\right]\bigg|_{s=\phi} = -7$$

$$f(t) = \left[-7 + 175t + 25e^{-t}\cos(7t + .23)\right]u(t)$$



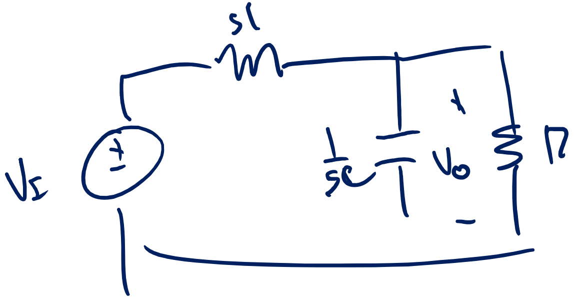
$$H(s) = \frac{R}{R + \frac{1}{sC}}$$

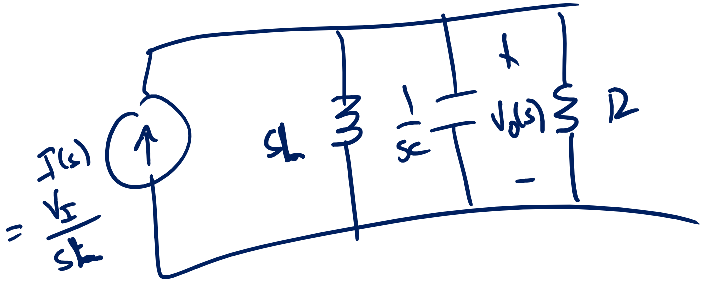


$$V_0 = H(s)\,V_I(s) + H_2(s)\cdot V_{IC}(s)$$

$$= \frac{(\text{Numer. of } H)(\text{Num. of } V_I)}{(\text{poles of } H)(\text{poles of } V_I)}$$

$$= \frac{k}{\text{pole of } H \#1} + \frac{k_n}{\text{pole of } H\#2} + \cdots + \frac{k_3}{\text{poles of } V_I} + \cdots$$

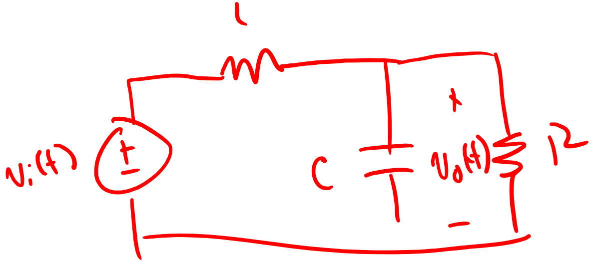




$$V_o = \frac{V_I}{sL}\left(sL \parallel \frac{1}{sC} \parallel R\right)$$

$$= \frac{V_I}{sL}\left(\frac{1}{sC + \frac{1}{sL} + \frac{1}{R}}\right)$$

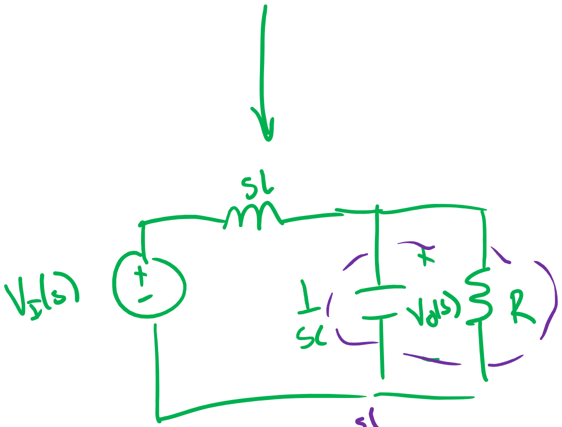
$$= V_I \frac{1}{s^2LC + 1 + s\frac{L}{R}}$$



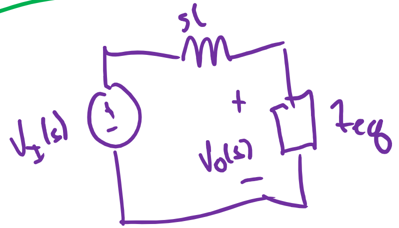
$$H(s) = \frac{\frac{1}{sC} \| R}{sL + \frac{1}{sC} \| R}$$

$$\frac{1}{sC} \| R = \frac{1}{\frac{1}{R} + sC} = \frac{R}{1 + sRC}$$

$$H(s) = \frac{\frac{R}{1+sRC}}{sL + \frac{R}{1+sRC}} = \boxed{\frac{R}{s^2 LCR + sL + R}} = \frac{1}{s^2 LC + s\frac{L}{R} + 1}$$



$$Z_{eq} = \frac{1}{sC} \| R$$

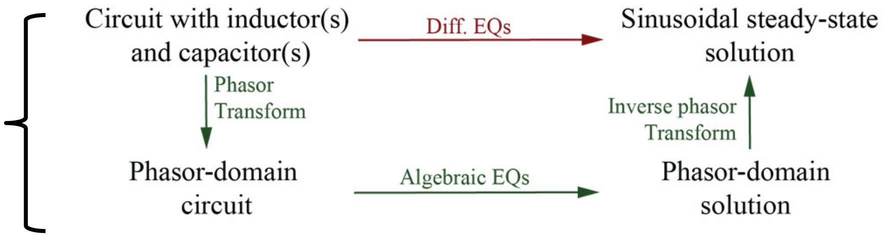


$$V_o(s) = V_I(s) \frac{Z_{eq}}{Z_{eq} + sL}$$

Transform Domains

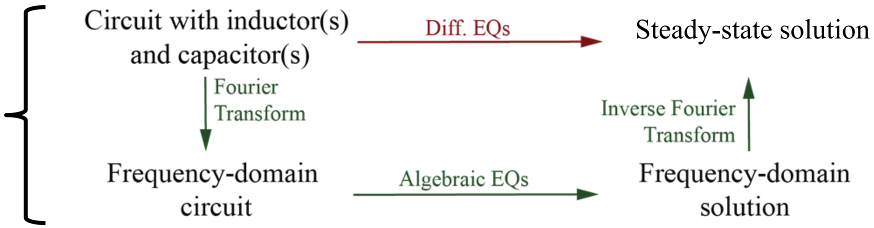
Phasor Transform

$$u(t) = A\cos(\omega_0 t + \varphi)$$



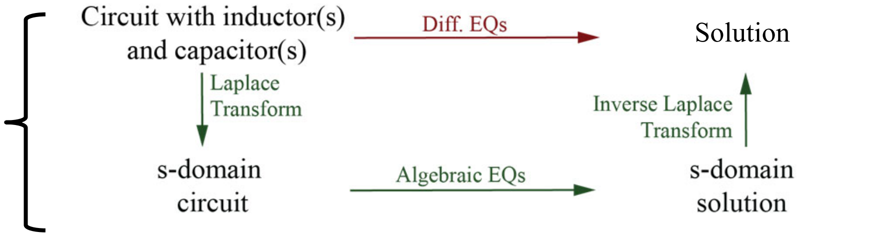
Fourier Transform

$$u(t) = \sum C_n \cos(n\omega_0 t + \varphi_n)$$

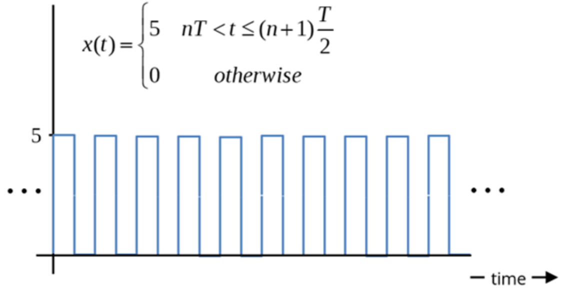
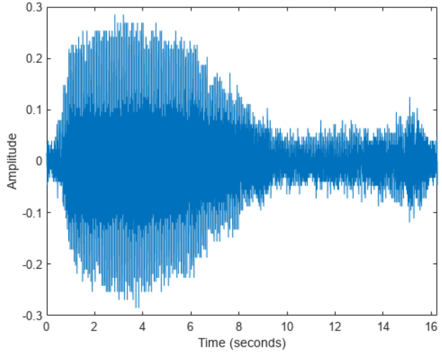
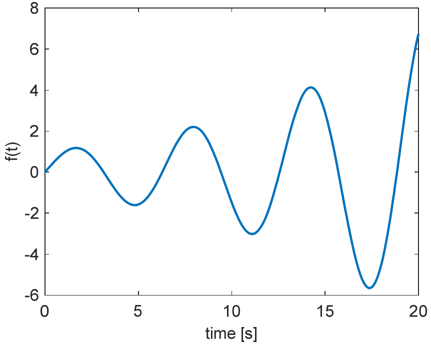
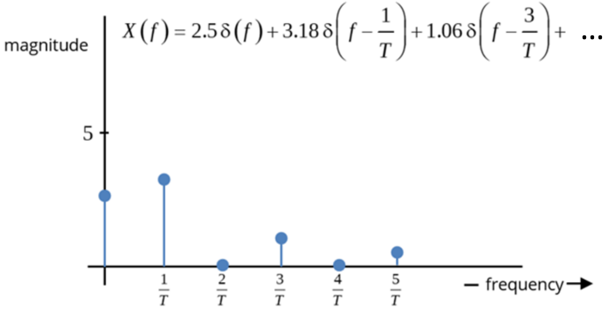
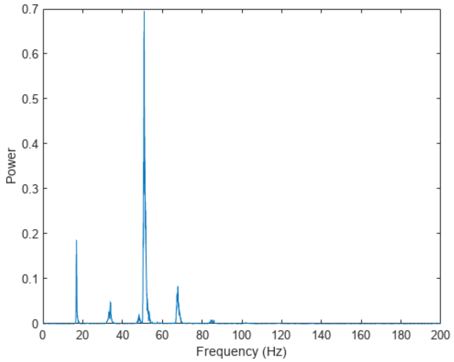
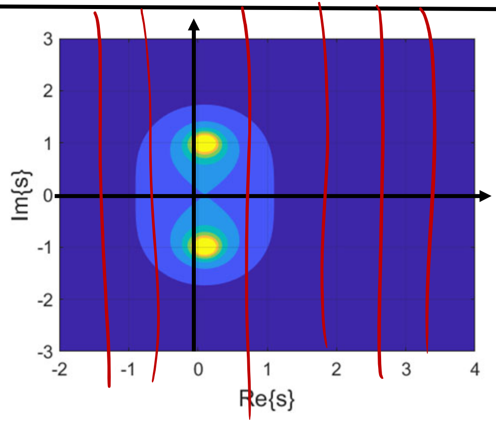


Laplace Transform

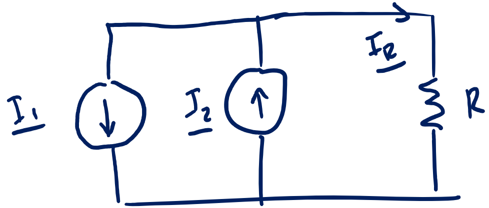
$$u(t) = \sum K_n e^{s_n t}$$



Transforms Visualized

	Fourier Series	Fourier Transform	Laplace Transform
Time Domain			
Frequency Domain			

Power Spectrum



$I_1 = I_2$ By inspection $I_R = \phi$

Therefore $P_R = \phi$

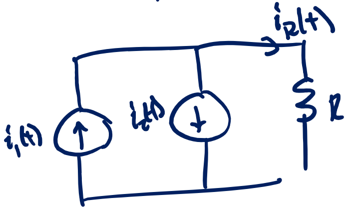
Correctly apply superposition:

$$I_R = I_{R1} + I_{R2} = (-I_1) + I_2 = \phi$$

Incorrect to apply superposition to power (because it's nonlinear)

If I did it anyway: $P_R = P_1 + P_2 = I_{1rms}^2 R + I_{2,rms}^2 R \neq \phi$ (wrong)

However, this will work if I have two sources at different frequencies



$i_1(t) = I_{A1}\cos(\omega_1 t)$ $\quad i_2(t) = I_{A2}\cos(\omega_2 t)$, $\quad \omega_1 \neq \omega_2$

$$P_R(t) = i_R(t)^2 R = (i_1(t) - i_2(t))^2 R$$

$$P_R = \frac{1}{T}\int_0^T i_R(t)^2 R\, dt = \frac{1}{T}\int_0^T (I_{A1}\cos(\omega_1 t) + I_{A2}\cos(\omega_2 t))^2 R\, dt$$

$$= \frac{R}{T}\int_0^T I_{A1}^2\cos^2(\omega_1 t) + I_{A2}^2\cos^2(\omega_2 t) + \underbrace{2 I_{A1} I_{A2}\cos(\omega_1 t)\cos(\omega_2 t)}_{I_{A1}I_{A2}\left(\cos(\omega_1 t + \omega_2 t) + \cos(\omega_1 t - \omega_2 t)\right)}\, dt$$

$\to \phi$ due to averaging

$$= I_{1rms}^2 R + I_{2rms}^2 R$$ (only for $\omega_1 \neq \omega_2$)

Limitations of Phasor Analysis

1. single frequency
2. sinusoids only
3. only steady-state response
4. LTI systems only

Remainder of the course: develop techniques to address 1 - 3

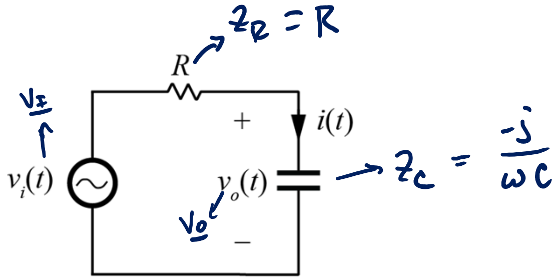
Approaches:

1. use superposition in the time-domain
   → Ch 15 or Frequency Response
2. Express arbitrary signal as a sum of (infinite) sinusoids
   → Ch 17 Fourier Series / Transform
3. Include exponentials with our sinusoids
   → Ch 14 Laplace Transform

Frequency Response

Phasor Analysis:

$$\underline{V_o} = \underline{V_I}\,\frac{Z_C}{Z_C + Z_R} = \underline{V_I}\,\frac{-j/\omega C}{-j/\omega C + R} = \underline{V_I}\left(\frac{1}{1 - j\omega CR}\right)$$



$$\underline{V_o} = \underline{V_I}\underbrace{\left(\frac{1}{1 - j\omega CR}\right)}_{\text{Frequency Response}} = \underline{V_I}\,\underbrace{H(j\omega)}_{\text{Frequency Response}}$$

→ tells us, at any ω, how does circuit alter input at the output

$$H(j\omega) = \frac{1}{1 - j\omega RC}$$

Gain: $|H(j\omega)| = \dfrac{1}{\sqrt{1^2 + (\omega RC)^2}}$

Phase: $\angle H(j\omega) = 0 - \tan^{-1}\left(\dfrac{\omega RC}{1}\right)$

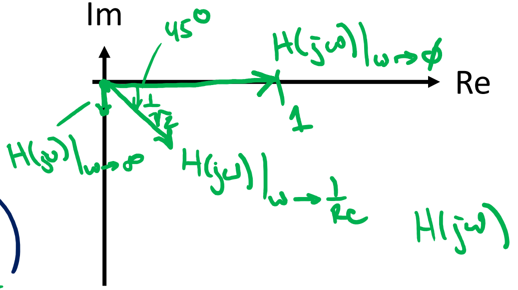
$= -\tan^{-1}(\omega RC)$

Any LTI circuit has $\underline{V_o} = \underline{V_I}\,H(j\omega)$

In polar form

$$V_{OA}\angle\varphi_{Vo} = (V_{IA}\angle\varphi_{Vin}) \cdot |H(j\omega)| \angle \varphi(H(j\omega))$$

$$= \underbrace{\left(V_{IA}\,|H(j\omega)|\right)}_{\text{Magnitudes multiply}} \angle \underbrace{\varphi_{Vin} + \varphi(H(j\omega))}_{\text{phases add}}$$



Bode Plot – Frequency Response

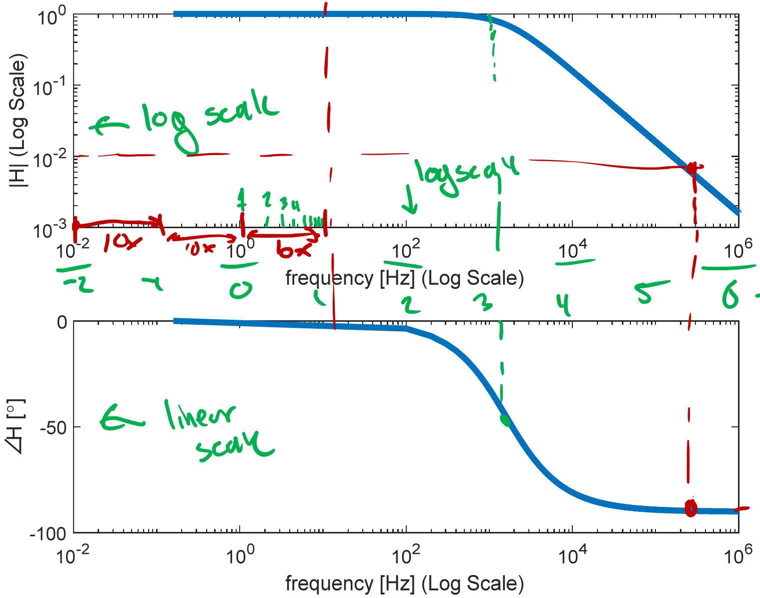
low-pass filter
(LPF)

$R = 10\Omega$
$C = 10nF$
$\frac{1}{RC} = 1.6kHz$

$V_o = V_I$ ← 1.6kHz →

Gain

phase



Fourier Series

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = \underline{a_0} + \sum_{k=1}^{\infty} \underline{a_k}\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)$$

$f(t)$ can be expressed this way if

1. $f(t)$ is single-valued
2. $\int_{t_0}^{t_0+T_0} |f(t)|\, dt$ exists
3. $f(t)$ had finite discontinuities and max/min per period

Need to find a_0, a_k, b_k for some function $f(t)$

for a_0: $\boxed{a_0 = \frac{1}{T}\int_0^{T_0} f(t)\,dt}$ a_0 is average / DC value of $f(t)$

For a_k:

$n \in \mathbb{Z}^+$ $a_n \to$ look at $\frac{1}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\,dt$

plugging in Fourier series for $f(t)$:

$$\frac{1}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\,dt = \frac{1}{T_0}\int_0^{T_0}\left[a_0 + \sum_{k=1}^{\infty} a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)\right]\cos(n\omega_0 t)\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \cancel{a_0\cos(n\omega_0 t)}^{\,0}\,dt + \frac{1}{T_0}\int_0^{T_0}\sum_{k=1}^{\infty}\left[a_k\cos(k\omega_0 t)\cos(n\omega_0 t) + b_k\sin(k\omega_0 t)\cos(n\omega_0 t)\right]dt$$

avg value of cos over n periods

$$= \frac{1}{T_0}\int_0^{T_0} \sum_{k=1}^{\infty} a_k \frac{1}{2}\Big(\cos\big(\cancel{(k+n)\omega_0 t}\big) + \underbrace{\cos\big((k-n)\omega_0 t\big)}_{\substack{=\phi \text{ if } k\neq n \\ =1 \text{ if } k=n}}\Big) + b_k \frac{1}{2}\Big(\cos\big(\cancel{(k+n)\omega_0 t - 90^\circ}\big) + \cancel{\cos\big((k-n)\omega_0 t - 90^\circ\big)}\Big)$$

$(k+n)$ period average $\rightarrow \phi$

$$\frac{1}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\,dt = \begin{cases} \phi & \text{if } k \neq n \\ \frac{a_n}{2} & \text{if } k = n \end{cases}$$

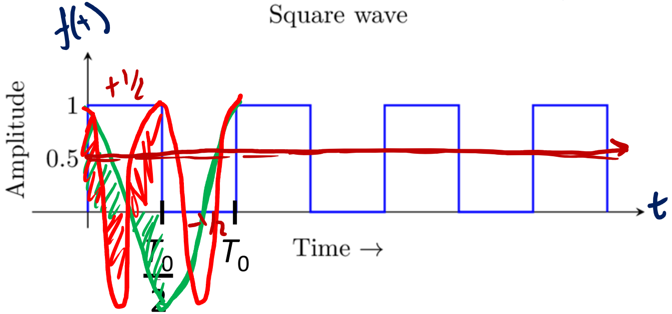
so,

$$\boxed{a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\,dt}$$

$$\boxed{b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\omega_0 t)\,dt}$$

Example Calculation

period : $T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$ $\qquad \omega_0 T_0 = 2\pi$



$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\,dt$$

$$= \frac{1}{T_0}\left[\int_0^{T_0/2} \underbrace{f(t)}_{=1 \text{ on } [0,T_0/2]}dt + \cancel{\int_{T_0/2}^{T_0} \underbrace{f(t)}_{=\phi \text{ on } [T_0/2,\,T_0]}dt}^{\,\phi}\right] = \boxed{a_0 = \frac{1}{2}}$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_0^{T_0/2}(1)\cdot\cos(n\omega_0 t)\,dt = \frac{2}{T_0}\left[\frac{1}{n\omega_0}\sin(n\omega_0 t)\right]\Bigg|_0^{T_0/2} = \underbrace{\frac{2}{T_0 n\omega_0}}_{\left(\frac{1}{n\pi}\right)}\left[\cancel{\underbrace{\sin\left(n\omega_0\frac{T_0}{2}\right)}_{\sin(n\pi)}}^{\,\phi} - \cancel{\sin(0)}^{\,\phi}\right]$$

$$\boxed{a_n = \phi \quad \forall n}$$

$$b_n = \frac{2}{T_0}\int_0^{T_0/2}(1)\sin(n\omega_0 t)\,dt = \frac{2}{T_0}\frac{1}{n\omega_0}\left[-\cos(n\omega_0 t)\right]\Bigg|_0^{T_0/2} = \frac{-1}{n\pi}\left[\underbrace{\cos(n\pi)}_{\pm 1} - \underbrace{\cos(0)}_{=+1}\right]$$

$$\boxed{b_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{2}{n\pi} & \text{for } n \text{ odd}\end{cases}}$$

Symmetry in Fourier Series

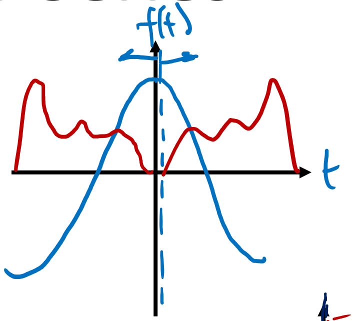
All apply with the DC component removed

Even functions

$f(t) = f(-t)$

Typo in the book in table 17.1

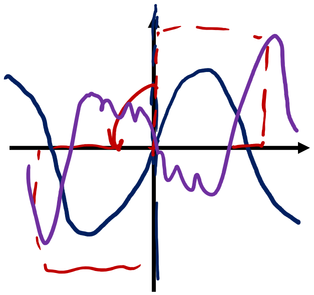
$$b_n = 0$$



Odd functions

$f(t) = -f(-t)$

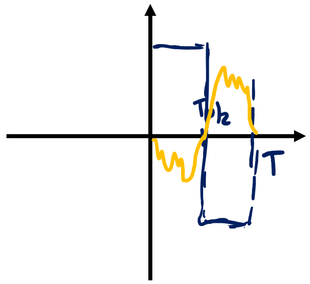
$$a_n = 0$$



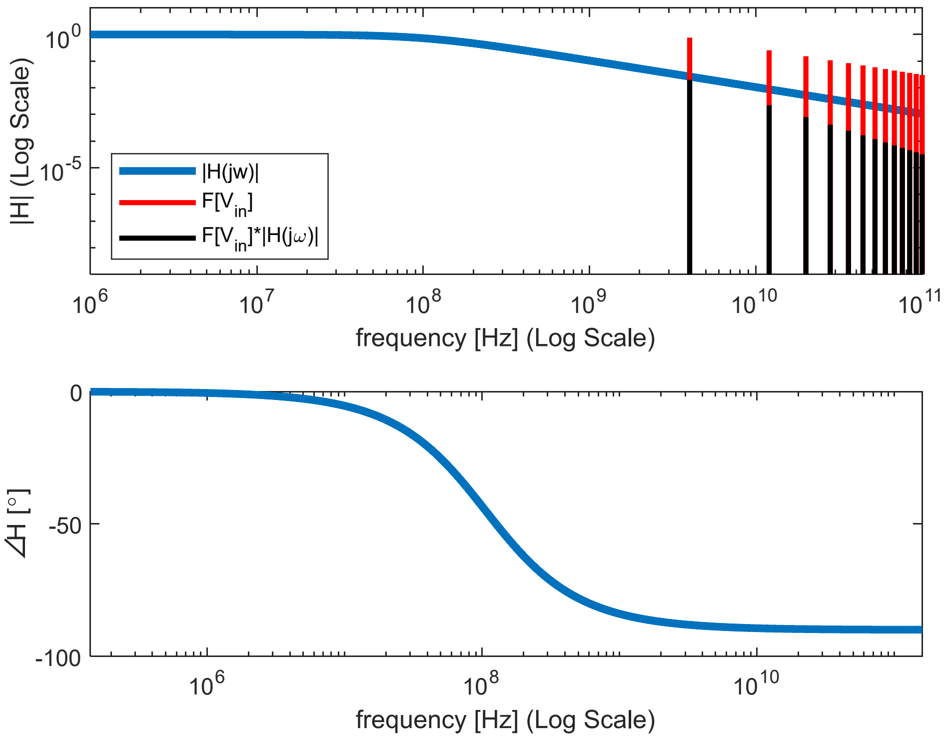
Half-wave symmetric functions

$f(t + \frac{T_0}{2}) = -f(t)$

$$a_n,\ b_n = 0 \text{ for even } n$$



Frequency Domain Interpretation



Fourier Series Representation

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$\boxed{a_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos(k\omega_0 t)\,dt} \qquad \boxed{b_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin(k\omega_0 t)\,dt}$$

$f(t)$ can be expressed this way if

1. $f(t)$ is single-valued
2. $\int_{t_0}^{t_0+T_0} |f(t)|\,dt$ exists
3. $f(t)$ had finite discontinuities and max/min per period

Alternate forms

$$f(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \varphi_k) \quad \begin{cases} A_k = \sqrt{{a_k}^2 + {b_k}^2} \\ \varphi_k = \tan^{-1}\left(\frac{b_k}{a_k}\right) \end{cases}$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \begin{cases} c_k = \frac{1}{2}(a_k - jb_k) \\ c_{-k} = \frac{1}{2}(a_k + jb_k) \\ c_0 = a_0 \end{cases}$$

$$\boxed{c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t}\,dt}$$

Non-periodic Waveforms: Fourier Transform

Fourier Series → works only for periodic waveforms

Fourier Transform → for non-periodic signals

Idea: treat any non-periodic signal as if it was periodic with $T \to \infty$

Fourier Series: $c_k = \frac{1}{T}\int_0^T f(t)\, e^{-jk\omega_0 t}\, dt$

Fourier Transform: $Tc_k = \boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt}$

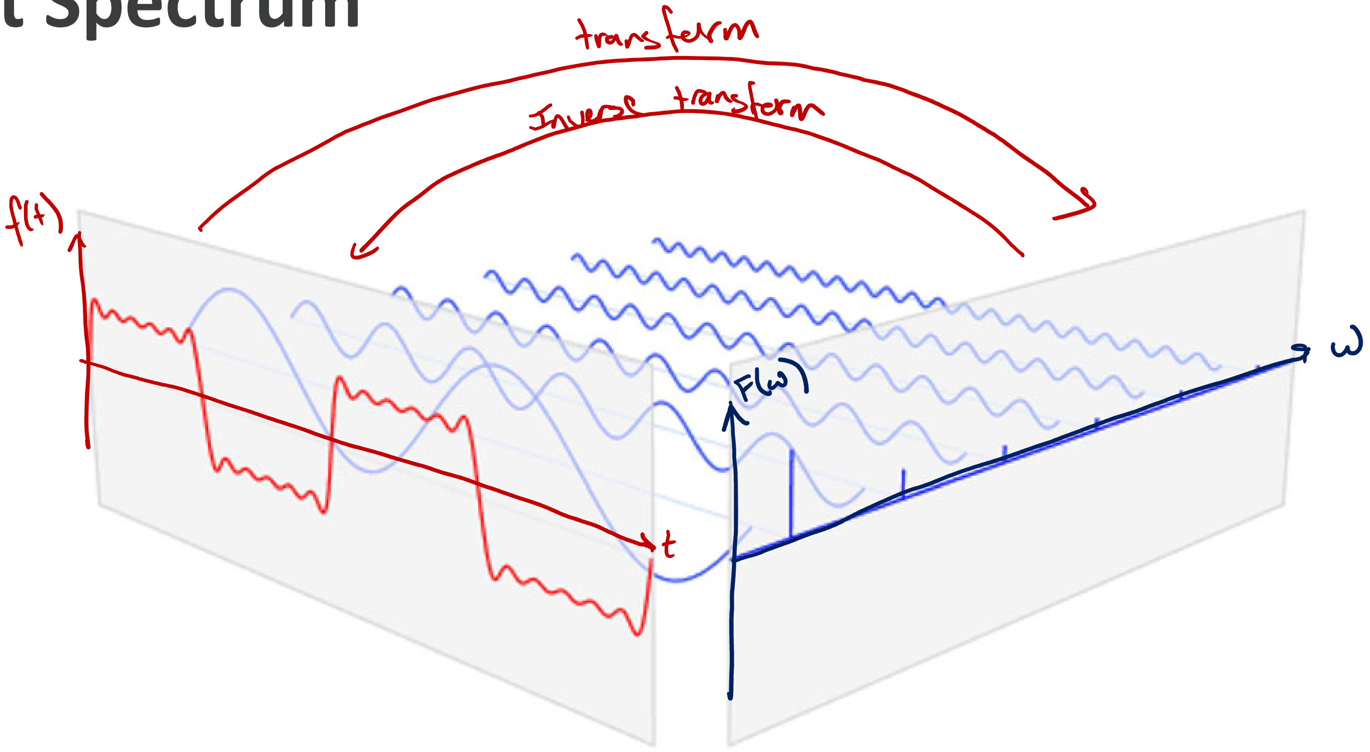
Fourier Series: summation $f(t) = \sum_{n=-\infty}^{\infty} c_k\, e^{jk\omega_0 t}$

Inverse Fourier Transform: $\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, dt}$

$f(t)$ can be expressed this way if

1. $f(t)$ is single-valued
2. $\int_{-\infty}^{\infty} |f(t)|\, dt$ exists
3. $f(t)$ had finite discontinuities and max/min in any closed interval

Input Spectrum



The Laplace Transform

Take Fourier Transform & replace $j\omega \rightarrow s = \sigma + j\omega$

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\,dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st}\,ds$$

σ_0 is any real number that works

Usually (always in ECE 202) we use Unilateral Laplace Transform ($f(t) = \emptyset$ for $t < 0$)

$$\boxed{F(s) = \int_{0^-}^{\infty} e^{-st} f(t)\,dt}$$

Laplace transform

$$\boxed{f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)\, e^{st}\,ds}$$

Inverse Laplace Transform

short-hand

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{f(t)\}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s)\}$$

$f(t) \longrightarrow F(s)$

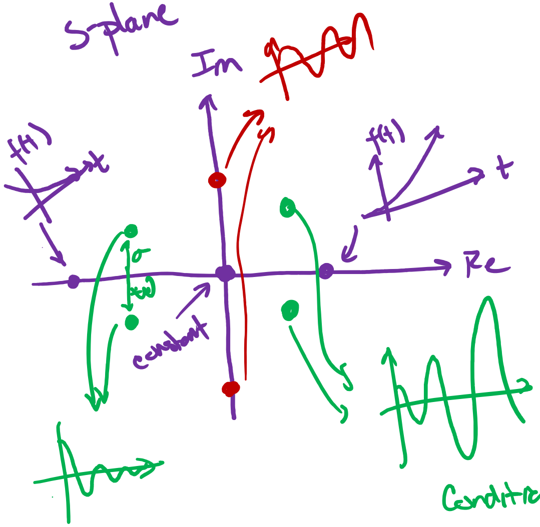
Time-domain		Frequency Domain		Laplace / s / complex Freq. Domain	
$f(t)$	ODEs	$F(j\omega)$	$H(j\omega)$	$F(s)$	$H(s)$
signals	systems	signals	systems	signals	systems

Complex Frequency

$s = \sigma + j\omega$ is a "complex frequency"

In Laplace domain our signal $f(t)$ is made up of a superposition of signals that look like $k_a e^{st} = k_a e^{(\sigma + j\omega)t} = k_a e^{\sigma t} e^{j\omega t}$

s-plane



if $\sigma = \phi$, $s = \phi + j\omega \rightarrow$ sinusoids

($s = -j\omega$ will always also show up)

if $\omega = \phi$ $s = \sigma + j\phi \rightarrow$ exponentials

converging if $\sigma < \phi$

if $\omega = \phi$ & $\sigma = \phi \rightarrow$ constants

if $\omega \neq \phi$ & $\sigma \neq \phi \rightarrow$ exponentials * sinusoids

$e^{\sigma t} \cos(\omega t + \phi)$

Conditions for Laplace Transform to exist

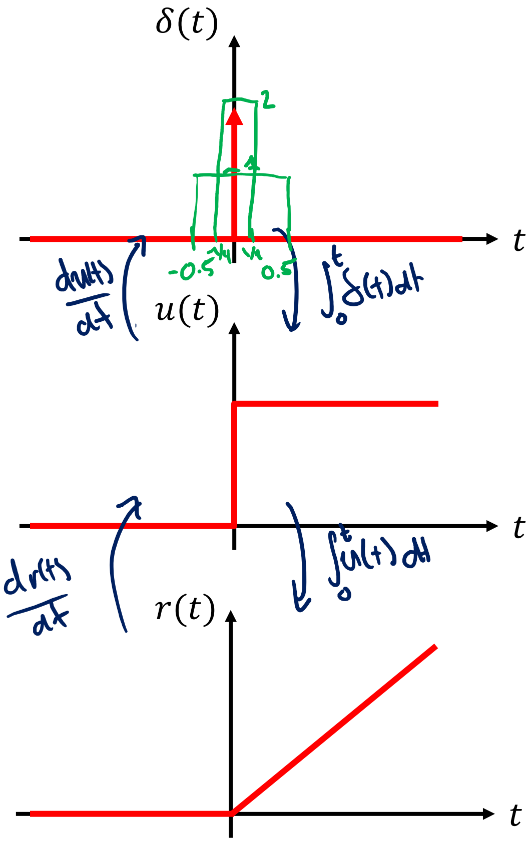
1. $f(t)$ is a function
2. $f(t)$ has a finite # of discontinuities & max/min over any finite time
3. $\int_0^\infty |e^{-at} f(t)| \, dt$ converges for some real a

Impulse, Step, and Ramp Functions

Impulse

step

ramp



$$\delta(t)\begin{cases}0 & t \neq 0 \\ \infty & t = 0\end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$

$$u(t)\begin{cases}0 & t < 0 \\ 1 & t > 0\end{cases}$$

$$r(t) = tu(t) = \begin{cases}0 & t \leq 0 \\ t & t \geq 0\end{cases}$$

Sifting Property of Impulse Function $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

so

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

Example Signal Laplace Transforms

$f(t) = u(t)$

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} e^{-st} u(t)\, dt = \int_0^{\infty} e^{-st}\, dt$$

$$= \left[\frac{-1}{s} e^{-st}\right]\Big|_{t=0}^{t=\infty} = \left[0 - \left(\frac{-1}{s}\right)\right]$$

$$\boxed{F(s) = \mathcal{L}\{u(t)\} = \frac{1}{s} \quad \text{if } Re\{s\} > 0}$$

Region of convergence for $\mathcal{L}\{u(t)\} \rightarrow Re\{s\} > 0$

$s = \sigma + j\omega \rightarrow \sigma > \emptyset$

$f(t) = e^{-at} u(t)$

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} e^{-at} u(t)\, dt = \int_0^{\infty} e^{-(s+a)t}\, dt$$

$$\boxed{\mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a} \quad \text{if } Re\{s+a\} > 0}$$

Generalize: $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

(where $F(s) = \mathcal{L}\{f(t)\}$)

Properties of the Laplace Transform

1. Uniqueness: if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\} = f(t)$

2. Linearity: $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} = F(s) + G(s)$

$$\mathcal{L}\{\alpha f(t)\} = \alpha F(s)$$

3. Differentiation:

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^{\infty} \underset{u}{e^{-st}} \underset{\frac{dv}{dt}}{\frac{df}{dt}}\, dt = \left(e^{-st} f(t)\right)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} (-s) e^{-st} f(t)\, dt$$

$$= \left(0 - f(0^-)\right) + s \underbrace{\int_{0^-}^{\infty} e^{-st} f(t)\, dt}_{F(s)}$$

$$\boxed{\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)}$$

Differentiation can be applied recursively

$$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s\left[sF(s) - f(0^-)\right] - f'(0^-) = s^2 F(s) - s f(0^-) - f'(0)$$

Initial and Final Value Theorems

Initial Value Theorem

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \boxed{\int_{0^-}^{\infty} e^{-st}\frac{df}{dt}\,dt = sF(s) - f(0^-)}$$

$$\lim_{s\to\infty}\left[\int_{0^-}^{0^+} \underbrace{e^{-st}}_{=1 \,@\, t=0}\frac{df}{dt}\,dt + \cancel{\int_{0^+}^{\infty} e^{-st}\frac{df}{dt}\,dt}\right] = \lim_{s\to\infty}\left[sF(s) - f(0^-)\right]$$

$$= f(0^+) - \cancel{f(0^-)} = \lim_{s\to\infty}\left[sF(s)\right] - \cancel{f(0^-)}$$

$$\boxed{= \lim_{t\to 0^+} f(t) = \lim_{s\to\infty}\left[sF(s)\right]}$$

fastest part of response

highest frequencies

Final Value Theorem :

$$\lim_{s\to 0}\left[\int_{0^-}^{\infty} e^{-st}\frac{df}{dt}\,dt\right] = \lim_{s\to 0}\left[sF(s) - f(0^-)\right]$$

$$f(t\to\infty) - \cancel{f(0^-)} = \lim_{s\to 0}\left[sF(s)\right] - \cancel{f(0^-)}$$

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left[sF(s)\right]}$$

All poles in LHP

(Final value is defined)

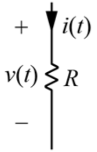
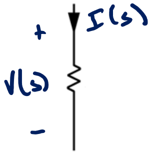
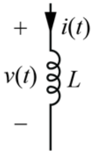
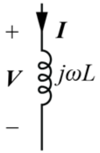
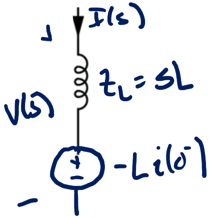
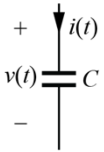
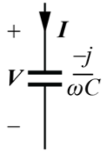
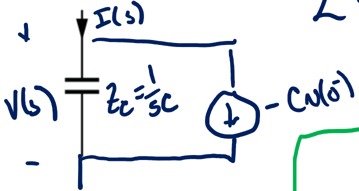
TABLE 14.2 Laplace Transform Operations

Operation	$f(t)$	$\mathbf{F}(s)$
Addition	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(s) \pm \mathbf{F}_2(s)$
Scalar multiplication	$kf(t)$	$k\mathbf{F}(s)$
Time differentiation	$\frac{df}{dt}$	$s\mathbf{F}(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2\mathbf{F}(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3\mathbf{F}(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
Time integration	$\int_{0^-}^{t} f(t)\,dt$	$\frac{1}{s}\mathbf{F}(s)$
	$\int_{-\infty}^{t} f(t)\,dt$	$\frac{1}{s}F(s) + \frac{1}{s}\int_{-\infty}^{0^-} f(t)\,dt$
Convolution	$f_1(t) * f_2(t)$	$\mathbf{F}_1(s)\mathbf{F}_2(s)$
Time shift	$f(t-a)u(t-a),\ a \geq 0$	$e^{-as}\mathbf{F}(s)$
Frequency shift	$f(t)e^{-at}$	$\mathbf{F}(s+a)$
Frequency differentiation	$tf(t)$	$-\frac{d\mathbf{F}(s)}{ds}$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^{\infty} \mathbf{F}(s)\,ds$
Scaling	$f(at),\ a \geq 0$	$\frac{1}{a}\mathbf{F}\left(\frac{s}{a}\right)$
Initial value	$f(0^+)$	$\lim_{s \to \infty} s\mathbf{F}(s)$
Final value	$f(\infty)$	$\lim_{s \to 0} s\mathbf{F}(s)$, all poles of $s\mathbf{F}(s)$ in LHP
Time periodicity	$f(t) = f(t + nT)$, $n = 1, 2, \ldots$	$\frac{1}{1 - e^{-Ts}}\mathbf{F}_1(s)$, where $\mathbf{F}_1(s) = \int_{0^-}^{T} f(t)e^{-st}\,dt$

TABLE 14.1 Laplace Transform Pairs

$f(t) = \mathscr{L}^{-1}\{\mathbf{F}(s)\}$	$\mathbf{F}(s) = \mathscr{L}\{f(t)\}$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!}u(t),\ n = 1, 2, \ldots$	$\frac{1}{s^n}$
$e^{-\alpha t}u(t)$	$\frac{1}{s + \alpha}$
$te^{-\alpha t}u(t)$	$\frac{1}{(s + \alpha)^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t),\ n = 1,2,\ldots$	$\frac{1}{(s + \alpha)^n}$
$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})u(t)$	$\frac{1}{(s + \alpha)(s + \beta)}$
$\sin \omega t\, u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t\, u(t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t + \theta)\, u(t)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
$\cos(\omega t + \theta)\, u(t)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$e^{-\alpha t}\sin \omega t\, u(t)$	$\frac{\omega}{(s + \alpha)^2 + \omega^2}$
$e^{-\alpha t}\cos \omega t\, u(t)$	$\frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$
$2\lvert k\rvert e^{\sigma t}\cos(\omega t - \angle k)\, u(t)$	$\frac{k}{s - (\sigma + j\omega)} + \frac{k^*}{s - (\sigma - j\omega)}$

Circuit Laplace Transform

Time Domain	Phasor Domain	s-Domain
 $v(t) = i(t)R$	 $\boldsymbol{V} = \boldsymbol{I}R$	
 $v(t) = L\dfrac{di}{dt}$	 $\boldsymbol{V} = j\omega L\boldsymbol{I}$	
 $i(t) = C\dfrac{dv}{dt}$	 $\boldsymbol{V} = \dfrac{-j}{\omega C}\boldsymbol{I}$	

$\mathcal{L}\{v(t)\} = \mathcal{L}\{i(t)R\}$

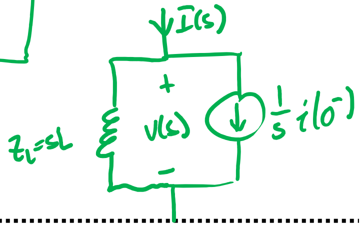
$V(s) = R\,I(s)$

$Z_R = R \rightarrow$ still called "Impedance"

$\mathcal{L}\{v(t)\} = \mathcal{L}\{L\frac{di(t)}{dt}\}$

$V(s) = sL\,I(s) - L\,i(0^-)$

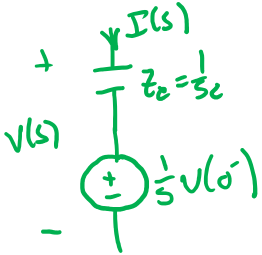
$I(s) = \frac{V(s)}{sL} + \frac{1}{s}i(0^-)$



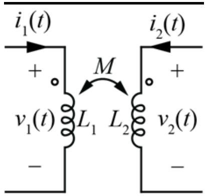
$\mathcal{L}\{i(t)\} = \mathcal{L}\{C\frac{dv(t)}{dt}\}$

$I(s) = sC\,V(s) - C\,v(0^-)$

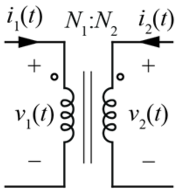
$V(s) = \frac{1}{sC}I(s) + \frac{1}{s}v(0^-)$



Time Domain

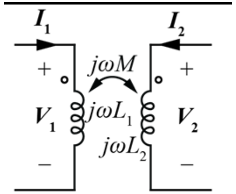


$$v_1(t) = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$
$$v_2(t) = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

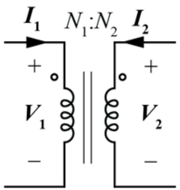


$$\frac{v_1(t)}{N_1} = \frac{v_2(t)}{N_2}$$
$$N_1 i_1(t) + N_2 i_2(t) = 0$$

Phasor Domain

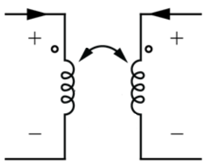


$$\boldsymbol{V}_1 = j\omega L_1 \boldsymbol{I}_1 + j\omega M \boldsymbol{I}_2$$
$$\boldsymbol{V}_2 = j\omega M \boldsymbol{I}_1 + j\omega L_2 \boldsymbol{I}_2$$

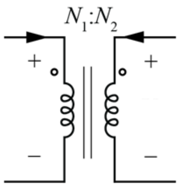


$$\frac{\boldsymbol{V}_1}{N_1} = \frac{\boldsymbol{V}_2}{N_2}$$
$$N_1 \boldsymbol{I}_1 + N_2 \boldsymbol{I}_2 = 0$$

s-Domain



$$V_1(s) = sL_1 I_1(s) - L_1 i_1(0^-) + sM I_2(s) - M i_2(0^-)$$
$$V_2(s) = sM I_1(s) - M i_1(0^-) + sL_2 I_2(s) - L_2 i_2(0^-)$$



$$\frac{V_1(s)}{N_1} = \frac{V_2(s)}{N_2}$$
$$N_1 I_1(s) + N_2 I_2(s) = 0$$

Laplace Transform of Diff EQs

N^{th} order circuit with sinusoidal input described by ($M \leq N$ for causality)

$$b_N \frac{d^N}{dt^N} v_o(t) + \cdots + b_1 \frac{d}{dt} v_o(t) + b_0 v_o(t) = a_M \frac{d^M}{dt^M} v_i(t) + \cdots + a_1 \frac{d}{dt} v_i(t) + a_0 v_i(t)$$

$$\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_o(t) = \sum_{i=0}^{M} a_i \frac{d^i}{dt^i} v_i(t)$$

Then the Laplace transform of the circuit, neglecting initial conditions, is

$$\mathcal{L}\left\{\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_o(t)\right\} = \mathcal{L}\left\{\sum_{i=0}^{M} a_i \frac{d^i}{dt^i} v_i(t)\right\}$$

Neglect Initial conditions

$$\sum_{i=0}^{N} b_i s^i V_o(s) = \sum_{i=0}^{M} a_i s^i V_i(s)$$

if we replace $s \to j\omega$ in $H(s)$ we get $H(j\omega)$, the frequency response

Rearranging:

Transfer function

$$\boxed{\frac{V_o(s)}{V_i(s)} = H(s) = \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i}}$$

Laplace Circuit Solution Algorithm

1. Transform all sources, signals into Laplace Domain
2. Transform circuit components (including initial conditions) into Laplace Domain
3. Solve the circuit using 201 techniques

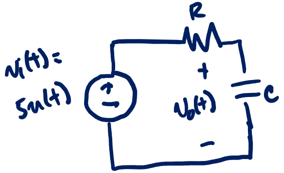
if multiple inputs

if any initial conditions

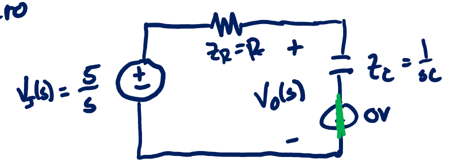
$$\boxed{V_o(s) = H(s)V_i(s) = \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i} V_i(s) + H_2(s)V_{i_2}(s)} + H_3(s)V_{IC1}(s) + \cdots$$

4. Inverse Laplace Transform to get back to time domain

Example Laplace Circuit Analysis



$v_o(t)$ @ $t \le 0$ is zero



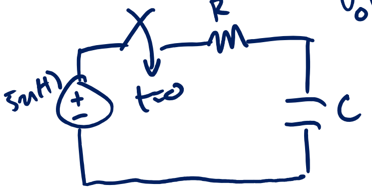
$v_o(t) = \left(5 - 5e^{-\frac{1}{RC}t}\right)u(t)$

$$V_o(s) = V_i(s)\frac{Z_C}{Z_C + Z_R} = \frac{5}{s}\,\frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{5}{s}\,\frac{\frac{1}{RC}}{\frac{1}{RC} + s}$$

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{\frac{1}{RC} + s}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{s} + \frac{-5}{\frac{1}{RC} + s}\right\}$$

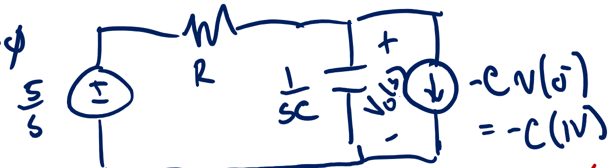
PFE (more review coming)

$$v(t) = \mathcal{L}^{-1}\left\{\frac{5}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-5}{\frac{1}{RC} + s}\right\} = \boxed{5u(t) - 5e^{-\frac{1}{RC}t}u(t)}$$



$V_o(t) = 1V$ for $t \le 0$

for $t > 0$



$$V_o(s) = \frac{5}{s}\left(\frac{\frac{1}{sC}}{R + \frac{1}{sC}}\right) + -(-C)\left(R \,\|\, \frac{1}{sC}\right) \longrightarrow C\left(\frac{1}{\frac{1}{R} + sC}\right) = \cancel{C}\,\frac{\frac{1}{\cancel{C}}}{\frac{1}{RC} + s} = \frac{1}{\frac{1}{RC} + s}$$

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = 5u(t) - 5e^{-\frac{1}{RC}t}u(t) + e^{-\frac{1}{RC}t}u(t)$$

Inverse Transforms

1. solve laplace domain circuit (for each input/IC source) to get some $V_o(s) = H(s)V_i(s)$

Transfer function circuit solution ↗ $\mathcal{L}\{v_i(t)\}$ ↑

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = \mathcal{L}^{-1}\{H(s)V_i(s)\}$$

this will look like $v_o(t) = \mathcal{L}^{-1}\left\{\frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i}\right\}$

some ratio of polynomials of s

usually, we'll need to factor $V_o(s)$ & do PFE

ex/ $V_o(s) = \frac{10}{s^2+4s+4} = \frac{10}{(s+2)(s+2)} = \frac{10}{(s+2)^2} \rightarrow v_o(t) = \mathcal{L}^{-1}\left\{\frac{10}{(s+2)^2}\right\} = 10te^{-2t}u(t)$

Factor 2nd order polynomial w/ quadratic formula

$$as^2 + bs + c = 0 \rightarrow r_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$a(s-r_1)(s-r_2) = 0$$

ex/ $I_o(s) = \frac{5s+1}{s+1} \rightarrow$ if $M \geq N$ (if highest exponent of s in numerator ≥ same in denom)

Use polynomial long division first

$$I_o(s) = \frac{5s+1}{s+1}$$

$$= 5 + \frac{-4}{s+1}$$

$$s+1 \overline{)\,5s+1\,}^{\;5}$$
$$-\underline{5s-5}$$
$$-4$$

Remainder

$$\mathcal{L}^{-1}\left\{5 + \frac{-4}{s+1}\right\} = \boxed{5\delta(t) - 4e^{-t}u(t) = i_o(t)}$$

Transfer Functions

factor

$$H(s) = \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i} = \frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}$$

polynomial form

factored pole/zero form

roots of numerator, z_i, are called zeros
- values of s at which $H(s) = 0$

roots of denominator, p_i, are called poles
- values of s near which $H(s) \to \infty$

if all a_i are real, then all z_i are either real or complex conjugate pairs
same is true for b_i & p_i

Both true for models of real circuits

Poles define the terms in $H(s) \to$ which become terms in $\mathcal{L}^{-1}\{V_I(s) \cdot H(s)\}$
zeros come into play in determining residues

Partial Fraction Expansion / Decomposition

$\underline{\underline{M < N}}$ (otherwise do long division first)

$$\underbrace{\frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i}}_{\text{polynomial form}} = \underbrace{\frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}}_{\text{factored pole-zero form}} \overset{\text{PFE}}{=} \underbrace{\frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_2)} + \cdots \frac{k_N}{(s-p_N)}}_{\text{Partial Fraction Expansion}}$$

k_i are called "residues"

if all p_i are real & distinct & $N > M$

→ then find k_i by "coverup" method → Multiply both sides by $(s-p_i)$ then evaluate at $s = p_i$

ex for k_2

$$\left.\frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)\cancel{(s-p_2)}\cdots(s-p_N)}\cancel{(s-p_2)}\right|_{s=p_2} = \left.\frac{k_1\cancel{(s-p_2)}}{\cancel{(s-p_1)}}\right|_{s=p_2} + \frac{k_2\cancel{(s-p_2)}}{\cancel{(s-p_2)}} + \cdots \left.\frac{k_N\cancel{(s-p_2)}}{\cancel{(s-p_N)}}\right|_{s=p_2}$$

ex/

$$F(s) = \frac{4(s+2)}{s^2+4s+3} \overset{\text{factor}}{=} \frac{4(s+2)}{(s+1)(s+3)} \overset{\text{PFE}}{=} \frac{k_1}{s+1} + \frac{k_2}{s+3} = \frac{2}{s+1} + \frac{2}{s+3}$$

$$k_1 = \left.\frac{4(s+2)}{(s+3)}\right|_{s=-1} = 2 = k_1$$

$$k_2 = \left.\frac{4(s+2)}{(s+1)}\right|_{s=-3} = 2 = k_2$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2e^{-t}u(t) + 2e^{-3t}u(t)$$

PFE: Repeated Roots

e.g. $F(s) = \dfrac{5s}{(s+2)^2} = \dfrac{k_1}{(s+2)} + \dfrac{k_2}{(s+2)^2} = \dfrac{k_1(s+2) + k_2}{(s+2)^2} = \dfrac{k_1 s + (k_2 + 2k_1)}{(s+2)^2}$

for $F(s) = \dfrac{N(s)}{(s-p_0)(s-p_1)^M} = \dfrac{k_0}{s-p_0} + \overbrace{\dfrac{k_1}{s-p_1} + \dfrac{k_2}{(s-p_1)^2} + \cdots}^{} + \dfrac{k_M}{(s-p_1)^M}$

k_0 can be found by coverup method

k_M can be found by coverup method

$k_1 \cdots k_{M-1}$ cannot use coverup method

find by $\begin{cases} \text{equating coefficients} \\ \text{differentiation} \end{cases}$

Repeated Roots: Equating Coefficients

ex/

$$F(s) = \frac{32s(s+1)}{(s+2)(s+10)^2} = \frac{k_1}{s+2} + \frac{k_2}{s+10} + \frac{k_3}{(s+10)^2}$$

$$k_1 = \left.\frac{32s(s+1)}{(s+10)^2}\right|_{s=-2} = 1$$

$$k_3 = \left.\frac{32s(s+1)}{(s+2)}\right|_{s=-10} = \frac{-320(-9)}{-8} = -360$$

Multiply both sides by full denominator & equate coeffients of powers of s

$$32s(s+1) = k_1(s+10)^2 + k_2(s+2)(s+10) + k_3(s+2)$$

$$32s^2 + 32s = k_1(s^2 + 20s + 100) + k_2(s^2 + 12s + 20) + k_3(s+2)$$

s^2 : $32 = k_1 + k_2 = 1 + k_2 \longrightarrow \boxed{k_2 = 31}$

s : $32 = k_1 20 + k_2 12 + k_3 = 20 + 12k_2 - 360$ ✓

s^0 : $\emptyset = k_1 100 + k_2 20 + 2k_3 = 100 + 20k_2 - 720$ ✓

$$F(s) = \frac{1}{s+2} + \frac{31}{s+10} + \frac{-360}{(s+10)^2} \longrightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = \left[1e^{-2t} + 31e^{-10t} - 360te^{-10t}\right]u(t)$$

Repeated Roots: Differentiation

$$\frac{1}{(s+p)^3} = \frac{k_1}{s+p} + \frac{k_2}{(s+p)^2} + \frac{k_3}{(s+p)^3}$$

$\frac{d^2}{ds^2}$ $\frac{d}{ds}$ coverup

ex

$$F(s) = \frac{32s(s+1)}{(s+2)(s+10)^2}$$

$k_1 = 1$, $k_3 = -360$ as before by coverup

Multiply both sides by repeated root with its full multiplicity, then take derivative(s) with respect to s before plugging in $s = p_i$

$$\frac{d}{ds}\left[\frac{32s(s+1)}{(s+2)\cancel{(s+10)^2}}\cancel{(s+10)^2}\right]\Bigg|_{s=-10} = \frac{d}{ds}\left[\frac{k_1}{s+2}(s+10)^2 + \frac{k_2}{\cancel{s+10}}\cancel{(s+10)^2} + \frac{k_3}{\cancel{(s+10)^2}}\cancel{(s+10)^2}\right]$$

$$\left[\frac{k_1 2\cancel{(s+10)}}{\cancel{s+2}}\ \phi + k_2 + \phi\right]\Bigg|_{s=-10}$$

$$k_2 = \frac{d}{ds}\left[\frac{32s(s+1)}{(s+2)}\right]\Bigg|_{s=-10} = \frac{(64s+32)(s+2) - (32s^2+32s)(1)}{(s+2)^2}\Bigg|_{s=-10} = 31$$

$$\boxed{k_2 = 31}$$

Complex Roots: Complex Math

ex $F(s) = \frac{1}{s^2-2s+2}$

$p_{1,2} = \frac{2 \pm \sqrt{4-4(1)(2)}}{2(1)} = 1 \pm j$

$F(s) = \frac{1}{s^2-2s+2} = \frac{1}{(s-(1+j))(s-(1-j))} = \frac{k_1}{(s-(1+j))} + \frac{k_2}{(s-(1-j))}$

complex roots will always occur in conjugate pairs $(s-p)(s-p^*)$ and their residues will always be complex conjugates $k_1 = k_2^*$, for any real signals & systems

$k_1 = \left.\frac{1}{s-(1-j)}\right|_{s=1+j} = \frac{1}{2j}$

$k_2 = \left.\frac{1}{s-(1+j)}\right|_{s=1-j} = \frac{-1}{2j} = k_1^*$ ✓

$$F(s) = \frac{\frac{1}{2j}}{s-(1+j)} + \frac{-\frac{1}{2j}}{s-(1-j)}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2j} e^{(1+j)t} u(t) + \left(\frac{-1}{2j}\right) e^{(1-j)t} u(t)$$

$$f(t) = \frac{1}{2j} e^{t} u(t) \left[e^{jt} - e^{-jt} \right]$$

$$f(t) = e^{t} \sin(t) u(t)$$

Euler's Formula

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

Complex Roots: General Case

$$\mathcal{L}^{-1}\left\{\frac{k}{s-(\sigma+j\omega)} + \frac{k^*}{s-(\sigma-j\omega)}\right\}$$

$$k = Re\{k\} + j\,Im\{k\}$$

$$k^* = Re\{k\} - j\,Im\{k\}$$

$$= \left[k e^{(\sigma+j\omega)t} + k^* e^{(\sigma-j\omega)t}\right]u(t)$$

$$= e^{\sigma t}\left[2Re\{k\}\left(\frac{e^{j\omega t}+e^{-j\omega t}}{2}\right) + (2j)j\,Im\{k\}\left(\frac{e^{j\omega t}-e^{-j\omega t}}{2j}\right)\right]u(t)$$

$$= e^{\sigma t}\left[2Re\{k\}\cos(\omega t) - 2Im\{k\}\sin(\omega t)\right]u(t)$$

$$= e^{\sigma t}\left[\sqrt{(2Re\{k\})^2+(2Im\{k\})^2}\,\cos\left(\omega t + \tan^{-1}\left(\frac{-2Im\{k\}}{2Re\{k\}}\right)\right)\right]u(t)$$

$$\boxed{= e^{\sigma t}\, 2|k| \cos(\omega t - \angle k)\, u(t)}$$

Complex Roots: Table Lookup

when possible, manipulate into terms already in the table

ex

$$F(s) = \frac{s+2}{s^2-2s+2} \underset{\text{complete the square}}{=} \frac{s+2}{(s-1)^2+1} = \frac{(s-1)+3}{(s-1)^2+1} = \frac{s-1}{(s-1)^2+1} + \frac{3(1)}{(s-1)^2+1}$$

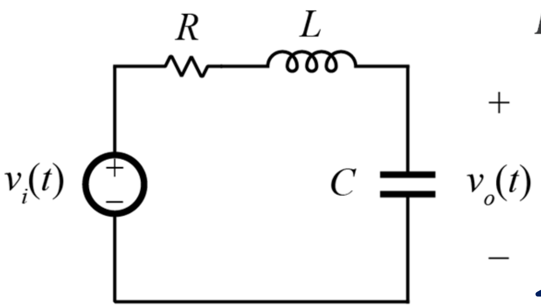
$$\mathcal{L}^{-1}\{F(s)\} = e^{t}\cos(t)u(t) + 3e^{t}\sin(t)u(t)$$

Example

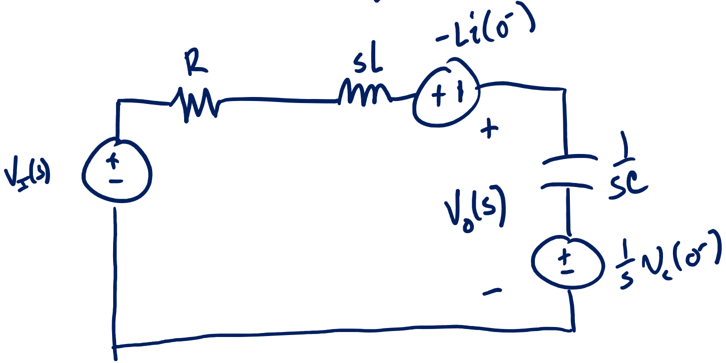
$$v_i(t) = \sin(2t)\,u(t)$$

$$L = 500\text{mH}, \quad C = 500\overset{m}{\cancel{\mu}}\text{F}, \quad R = 2\Omega$$

$$v_c(0) = 5\text{V}, \quad i_L(0) = -2\text{A}$$



$\mathcal{L}\{\text{circuit}\}$



$$V_I(s) = \mathcal{L}\{v_i(t)\} = \frac{2}{s^2+4}$$

Solve circuit in s-domain:

$$V_o(s) = H_I(s)\,V_I(s) + H_L(s)\left(-Li(0^-)\right) + H_C(s)\left(\tfrac{1}{s}v_c(0^-)\right)$$

$$= H_I(s)\,V_I(s) + \left(-Li(0^-)\right)\left(-H_I(s)\right) + \left(\tfrac{1}{s}v_c(0^-)\right)\left(1 - H_I(s)\right)$$

$$H_I(s) = \frac{\frac{1}{sC}}{\frac{1}{sC} + sL + R} = \frac{1}{s^2LC + sCR + 1}$$

$$H_I(s) = \frac{1}{s^2LC + sCR + 1} = \frac{1}{\frac{s^2}{4} + s + 1} = \frac{4}{s^2 + 4s + 4} = \frac{4}{(s+2)^2}$$

$$V_I(s) = \frac{2}{s^2+4} = \frac{2}{(s+2j)(s-2j)}$$

$$V_o(s) = \underbrace{\frac{4}{(s+2)^2}\frac{2}{s^2+4}}_{1} + \underbrace{(-1)\frac{4}{(s+2)^2} + \frac{5}{s}}_{\text{Look up in table}} - \underbrace{\frac{5}{s}\frac{4}{(s+2)^2}}_{2}$$

1 PFE: $$\frac{4}{(s+2)^2}\frac{2}{\underset{(s^2+4)}{(s+2j)(s-2j)}} = \frac{k_1}{s+2} + \frac{k_2}{(s+2)^2} + \frac{k_3}{s+2j} + \frac{k_3^*}{s-2j}$$

$$k_2 = \left.\frac{8}{s^2+4}\right|_{s=-2} = 1$$

$$k_3 = \left.\frac{8}{(s+2)^2(s-2j)}\right|_{s=-2j} = \frac{8}{(\cancel{-4} - 8j + \cancel{4})(-4j)} = \frac{-1}{4} = k_3^*$$

$$k_1 = \frac{d}{ds}\left[\frac{8}{s^2+4}\right]\Bigg|_{s=-2} = \left[8(-1)(s^2+4)^{-2}(2s)\right]\Bigg|_{s=-2} = \frac{-8}{8^2}(-4) = \frac{1}{2}$$

2

$$\frac{5}{s}\,\frac{4}{(s+2)^2} = \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{k_3}{(s+2)^2}$$

$$k_1 = \left.\frac{20}{(s+2)^2}\right|_{s=0} = 5$$

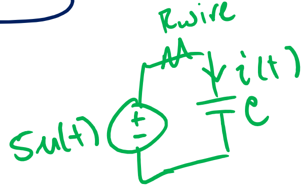
$$k_2 = \frac{d}{ds}\left[\frac{20}{s}\right]\Bigg|_{s=-2} = \left.\frac{-20}{s^2}\right|_{s=-2} = -5$$

$$k_3 = \left.\frac{20}{s}\right|_{s=-2} = -10$$

$$V_o(s) = \frac{1/2}{s+2} + \frac{1}{(s+2)^2} + \frac{-1/4}{s+2j} + \frac{-1/4}{s-2j} + \frac{-4}{(s+2)^2} + \cancel{\frac{5}{s} - \frac{5}{s}} - \frac{-5}{s+2} - \frac{-10}{(s+2)^2}$$

$$V_o(s) = \frac{5+1/2}{s+2} + \frac{7}{(s+2)^2} + \frac{-1/4}{s+2j} + \frac{-1/4}{s-2j}$$

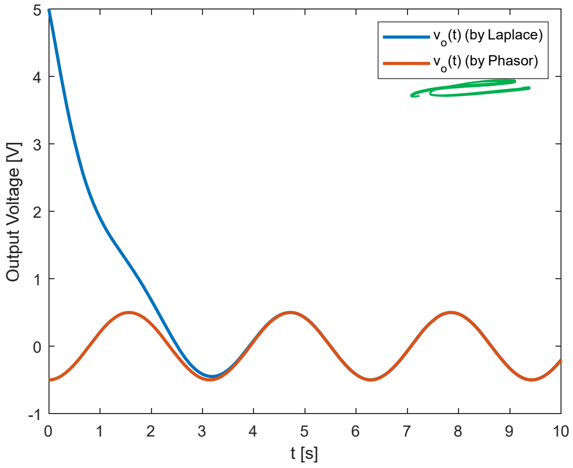
$$\boxed{v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = \left[5.5e^{-2t} + 7te^{-2t} + 2\tfrac{1}{4}\cos(2t - 0^\circ)\right]u(t)}$$

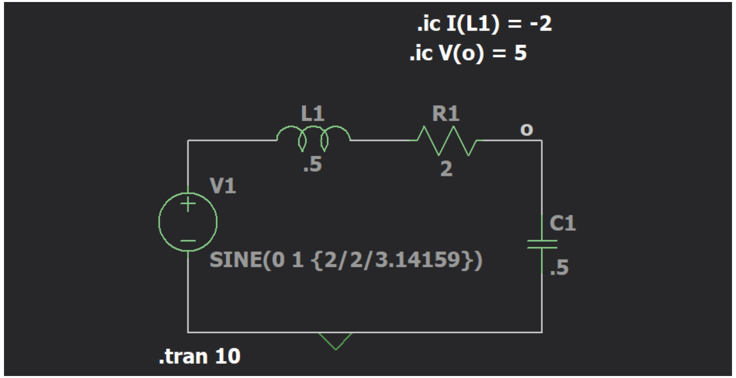


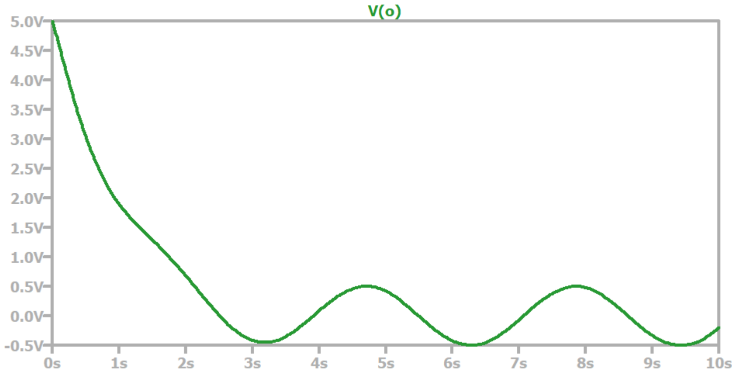
Comparison to Simulation

Phasor → $v_i(t) = \sin(2t)$ & neglect ICs } sinusoidal steady-state

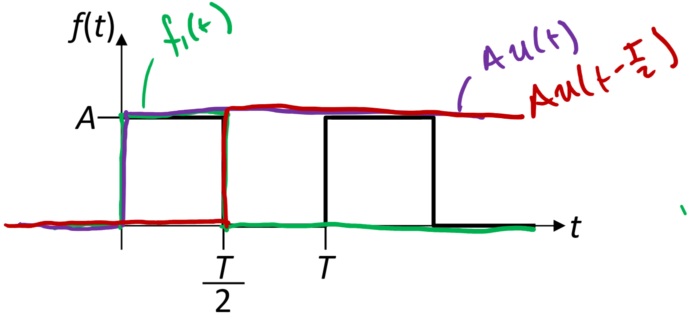
Laplace → $v_i(t) = \sin(2t)\,u(t)$ & include ICs } 'Any signal' transient & steady-state







Laplace Transform of Periodic PWL Signals



$$f(t) = \begin{cases} A & kT+0 < t < \frac{T}{2}+kT \\ \emptyset & kT+\frac{T}{2} < t < T+kT \end{cases} \qquad k \in \mathbb{Z}^+$$

$$f_1(t) = Au(t) - Au(t-\tfrac{T}{2}) \rightarrow \text{first period only}$$

$$f(t) = \sum_{k=0}^{\infty} f_1(t-kT)$$

Laplace transform:

$$\mathcal{L}\{f_1(t)\} = F_1(s) = A\frac{1}{s} - Ae^{-s\frac{T}{2}}\frac{1}{s} = \frac{A}{s}\left(1-e^{-s\frac{T}{2}}\right)$$

$$\mathcal{L}\{f(t)\} = F(s) = \sum_{k=0}^{\infty} e^{-sTk}F_1(s) = \sum_{k=0}^{\infty} e^{-sTk}\left(\frac{A}{s}\left(1-e^{-s\frac{T}{2}}\right)\right) = \frac{A}{s}\left(1-e^{-s\frac{T}{2}}\right)\cdot\sum_{k=0}^{\infty}\left(e^{-sT}\right)^k$$

$$r = e^{-sT}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad |r|<1$$

$$\boxed{F(s) = \frac{\frac{A}{s}\left(1-e^{-s\frac{T}{2}}\right)}{\left(1-e^{sT}\right)} = \frac{F_1(s)}{1-e^{-sT}}}$$

Pole Locations

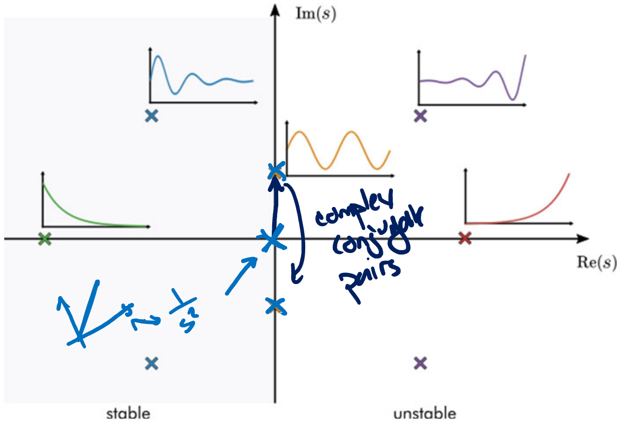
$$\boxed{x = N_H + N_I}$$

$$V_o(s) = V_I(s)H(s) = \left(\frac{\sum_{i=0}^{M_I} c_i s^i}{\sum_{i=0}^{N_I} d_i s^i}\right)\left(\frac{\sum_{i=0}^{M_H} a_i s^i}{\sum_{i=0}^{N_H} b_i s^i}\right) = \frac{(s-z_1)(s-z_2)\cdots(s-z_{M_H+M_I})}{(s-p_1)(s-p_2)\cdots(s-p_{N_H+N_I})}$$

$$V_o(s) = \frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_2)} + \cdots + \frac{k_x}{(s-p_x)}$$

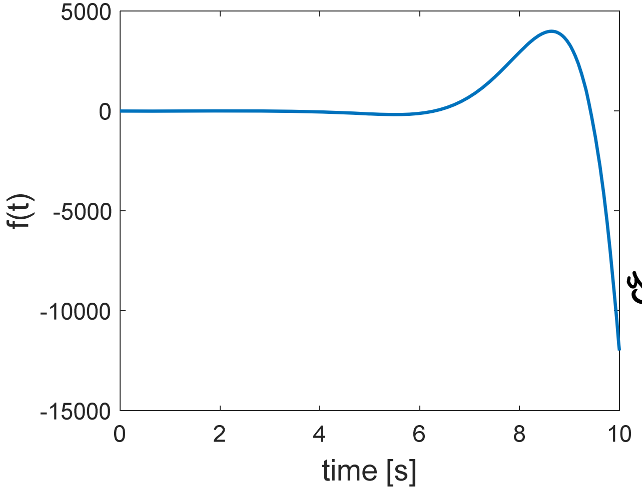
poles of both $V_I(s)$ & $H(s)$ determine the "type" of signals in the output

$$v_o(t) = k_1 e^{p_1 t} u(t) + k_2 e^{p_2 t} u(t) + \cdots$$

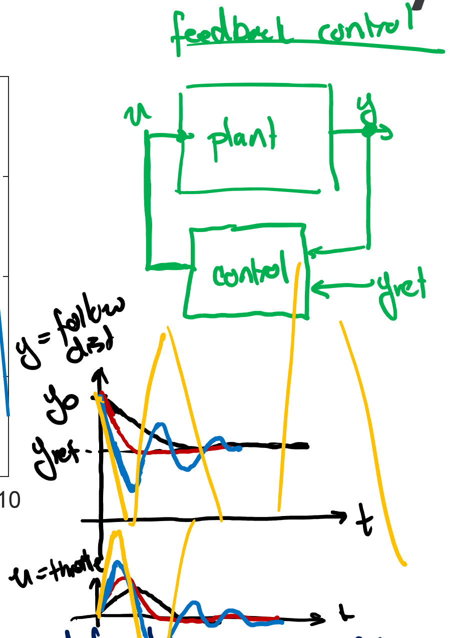


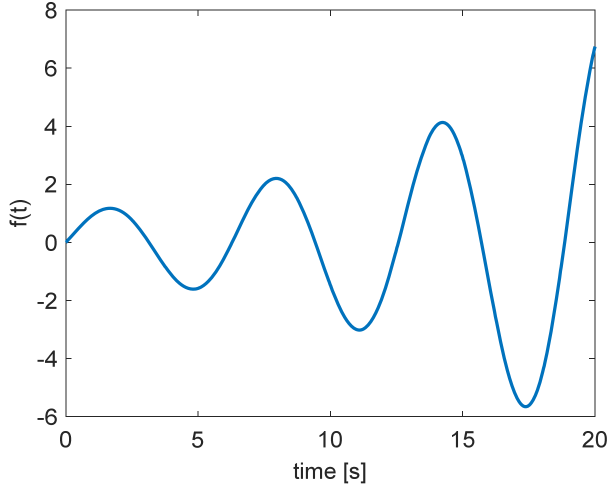
Output has poles/terms from both $H(s)$ & $V_I(s)$

Unbounded Signals & Unstable Systems



$f(t) = e^{t}\sin(t)$





$f(t) = e^{t/10}\sin(t)$

Bounded signals → mathematical definition $f(t)$ is bounded iff $\exists B$ s.t. $|f(t)| < B \ \forall t$

BIBO stability → "Bounded input, Bounded output" stability

Always want BIBO stable circuits / $H(s)$

Laplace and Fourier Revisited

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} f(t)\, dt$$

$s \to j\omega \quad (\sigma = 0)$

Laplace Transform (Bilateral):

$$F(s) = \int_{-\infty}^{+\infty} e^{-st} f(t)\, dt$$

$s = \sigma + j\omega$

If we let $s \to j\omega$ the laplace transform & fourier transform are equivalent, but for some signals the integrals won't converge

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

$s \to j\omega$

Inverse Laplace Transform (Bilateral):

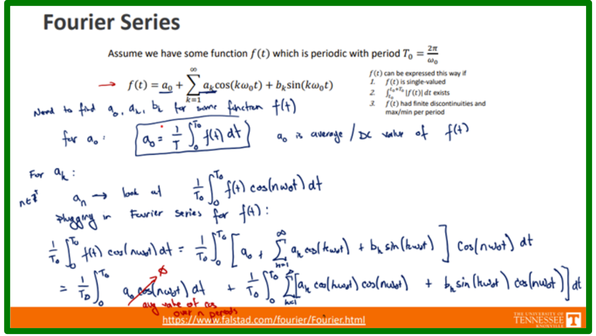
$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st} ds$$

Laplace Explanation

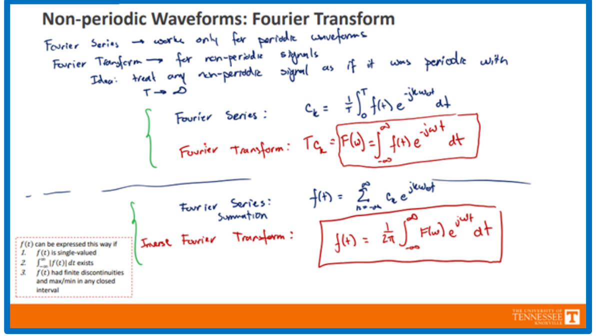
$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t)\, dt = \int_{0^-}^{+\infty} e^{-\sigma t} e^{-j\omega t} f(t) dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{\sigma t} e^{j\omega t} F(s)\, ds$$

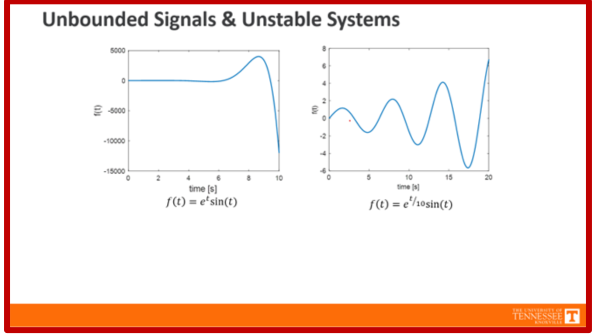
L21



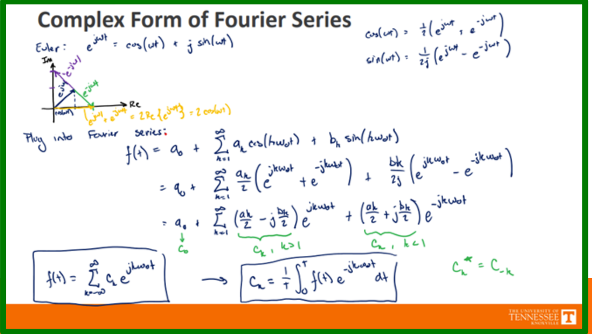
L24



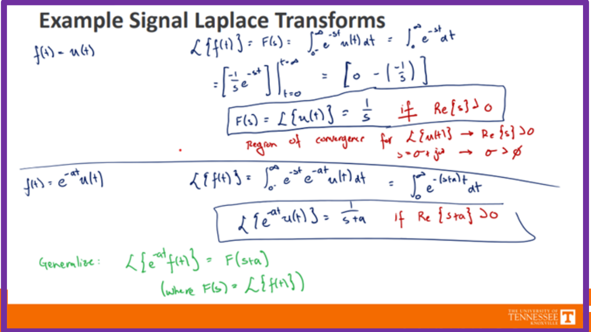
L30



L23



L25

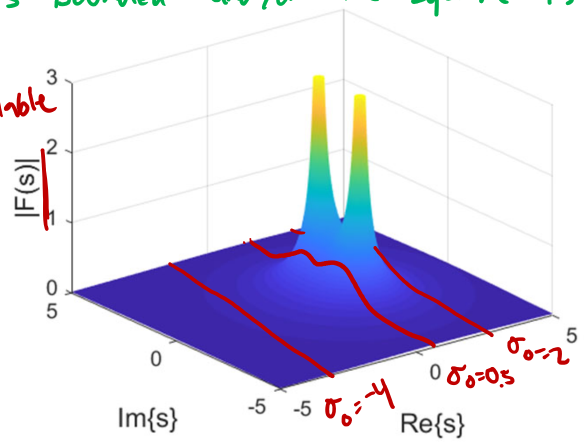


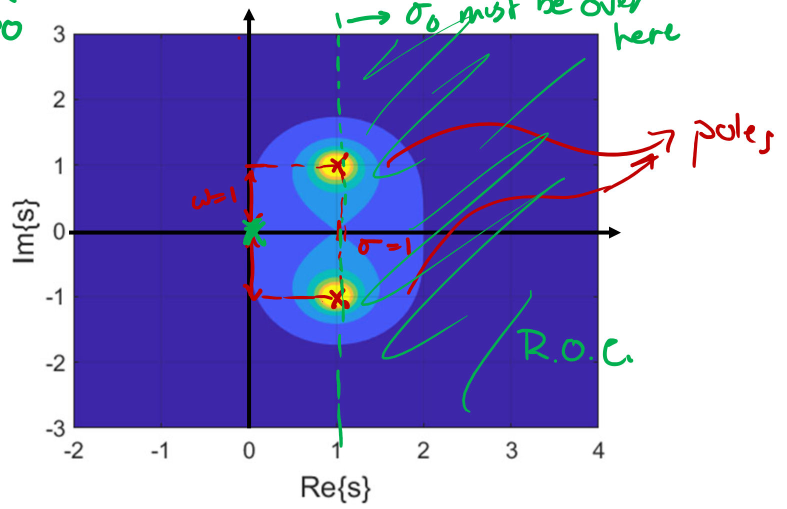
The s-plane

The Region of convergence of a Laplace transform is the complex plane to the right of all poles

If all poles are in the open left half-plane, (LHP) the signal is bounded and/or the system is BIBO stable

RHP poles $\Leftrightarrow$ Unstable





$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s - (1 + j))(s - (1 - j))}$$

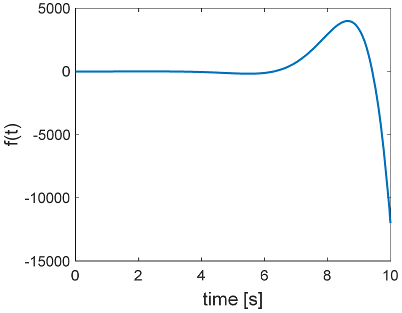
ROC is $Re\{s\} > 1$

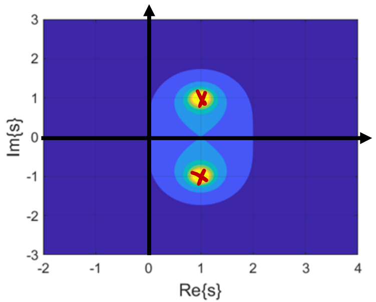
$f(t) = e^t \sin(t)$

$F(s) = \int_{0^-}^{\infty} \underbrace{e^{-st} e^{t}} \sin(t)\, dt \rightarrow$ Need $Re\{s\} > 1$ for convergence

Example Functions

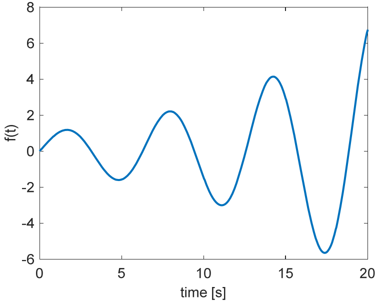
$$f(t) = e^{t}\sin(t)$$

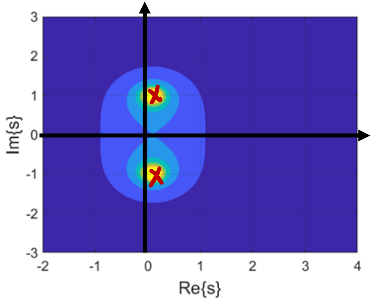




$$F(s) = \frac{1}{(s-(1+j))(s-(1-j))}$$

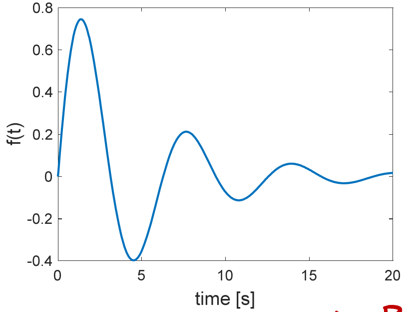
$$f(t) = e^{t/10}\sin(t)$$



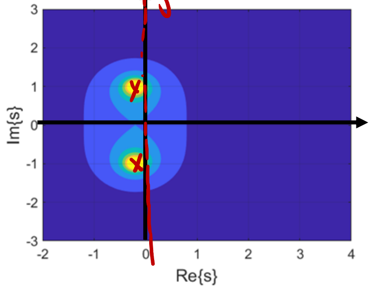


$$F(s) = \frac{1}{\left(s-\left(\frac{1}{10}+j\right)\right)\left(s-\left(\frac{1}{10}-j\right)\right)}$$

$$f(t) = e^{-t/5}\sin(t)$$



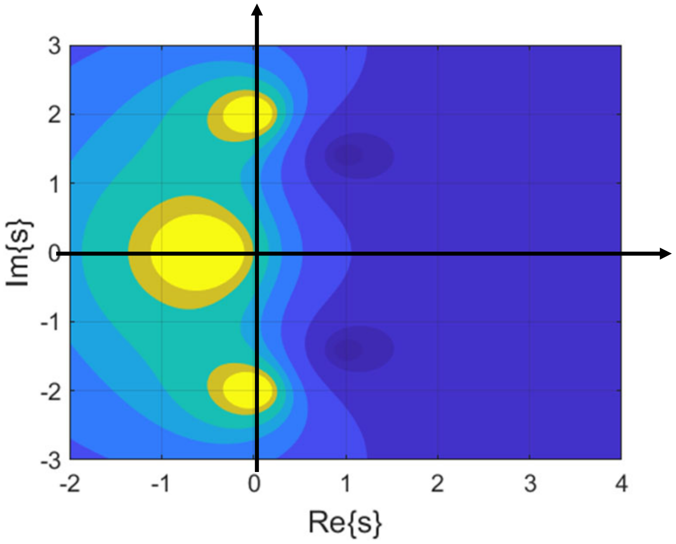
jw axis in ROC

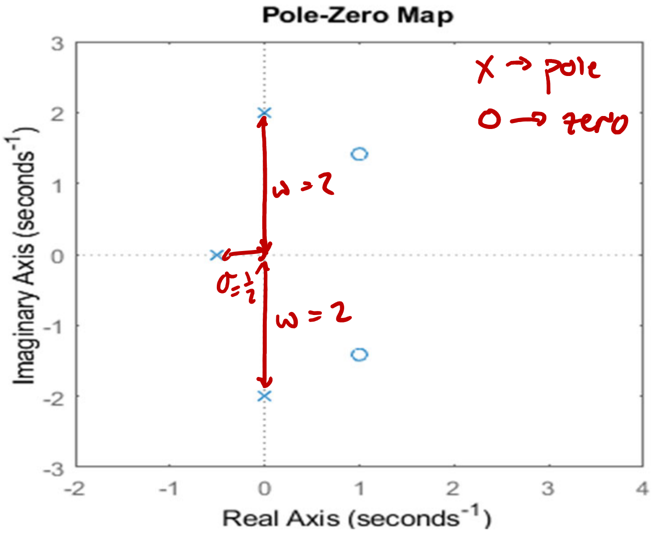


$$F(s) = \frac{1}{\left(s+\left(\frac{1}{5}+j\right)\right)\left(s+\left(\frac{1}{5}-j\right)\right)}$$

Pole-Zero Map

$$F(s) = -\frac{\left(s + \left(-1 + j\sqrt{2}\right)\right)\left(s + \left(-1 - j\sqrt{2}\right)\right)}{\left(s + {}^{1}/_{2}\right)(s + j2)(s - j2)}$$





MATLAB:

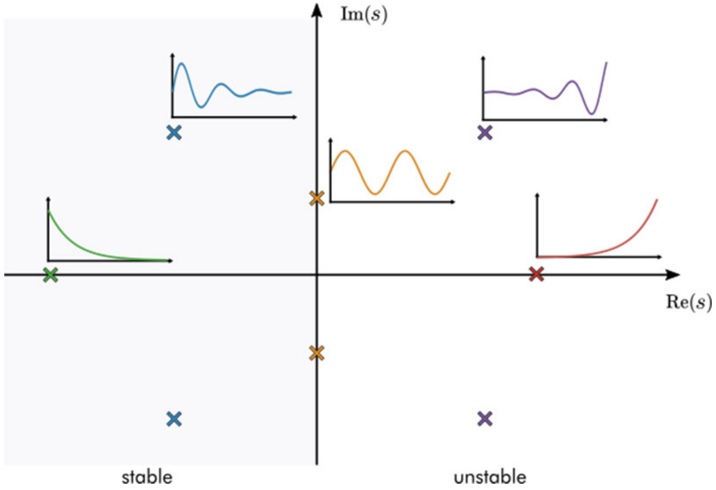
```
[R,X] = meshgrid(-2:.01:4,-3:.01:3);
s = R + 1j*X;
Fs = (s.^2-2*s+3)./(s.^2 + 4)./(s + 1/2);
[C,h] = contourf(R,X,abs(Fs));


h.LevelList = [0 .05 .1 .25 .5 1 1.5 2];
h.LineStyle = 'none';
```

MATLAB:

```
s = tf('s');
Fs = 2/(s^2 + 4) - 1/(s + 1/2);
pzmap(Fs);
```

Poles-Zero Plot



Takeaways:

1. Pole location tells us the "form" of our function

2. Complex poles/zeros & their residues always show up as conjugate pairs (for real signals/systems)

3. If all poles are in the open LHP
   - signal is bounded
   - system/circuit is BIBO stable

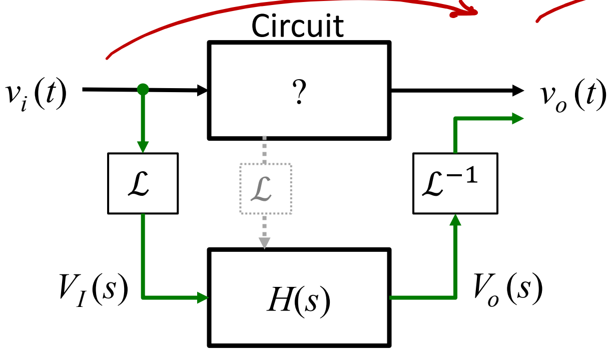
   If any pole is in RHP → unbounded unstable

   If pole(s) on $j\omega$-axis, need to look at multiplicity

4. If all poles in open LHP, $j\omega$-axis is in region of convergence $H(s \to j\omega)$ is Freq. Resp.

System I/O Relationship

201 approach $\rightarrow$ solve Diff EQs



$\mathcal{L}\{v_i(t)\} = V_I(s)$

Take the Laplace transform of the circuit & solve it to get $H(s)$

$V_o(s) = H(s)\,V_I(s)$

$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\}$

$v_o(t) = \mathcal{L}^{-1}\{V_I(s)\,H(s)\}$

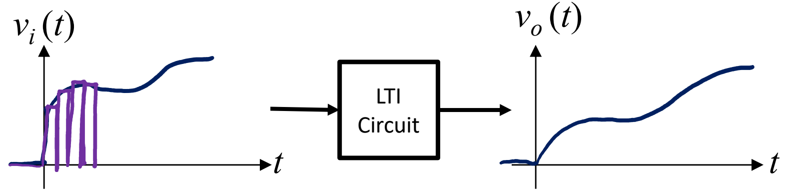
what is $\mathcal{L}^{-1}\{H(s)\}$?

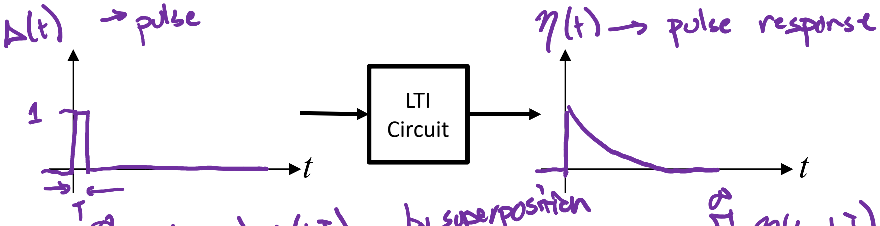
$\boxed{h(t) \rightarrow \text{impulse response of circuit}}$

Look at what happens if $v_i(t) = \delta(t)$? $\rightarrow$ $\mathcal{L}\{\delta(t)\} = 1$

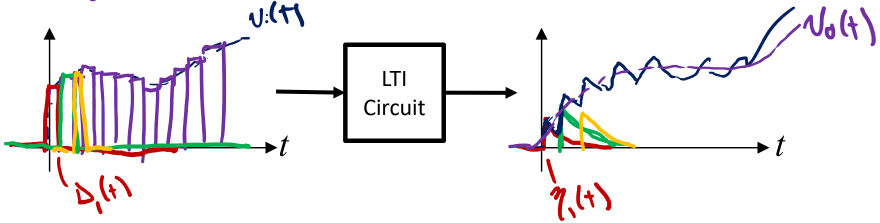
the $V_o(s) = H(s)\,(1)$ & $v_o(t) = h(t)$

Convolution





$$v_i(t) = \sum_{k=0}^{\infty} \Delta(t-kT)\, v_i(kT) \xrightarrow{\text{by superposition}} v_o(t) = \sum_{k=0}^{\infty} \eta(t-kT)\, v_i(kT)$$



Now, let $T \rightarrow 0$

$$v_i(t) = \int_0^{\infty} \delta(t-\tau)\, v_i(\tau)\, d\tau$$

$\downarrow$

by sifting property of $\delta(t)$

$$\boxed{v_o(t) = \int_0^{\infty} h(t-\tau)\, v_i(\tau)\, d\tau}$$

convolution integral

The Convolution Integral

$$v_o(t) = \int_0^\infty h(t-\tau)\, v_i(\tau)\, d\tau = \int_0^\infty v_i(t-\tau)\, h(\tau)\, d\tau = \underbrace{v_i(t) * h(t)}_{\text{short-hand}} = h(t) * v_i(t)$$

formally convolution integral $y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau$

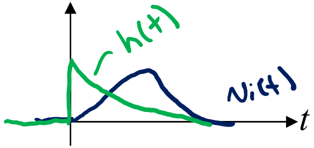
output

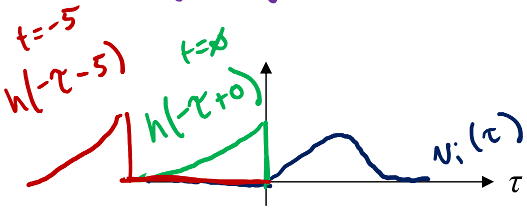
input (not necessarily step fcn)

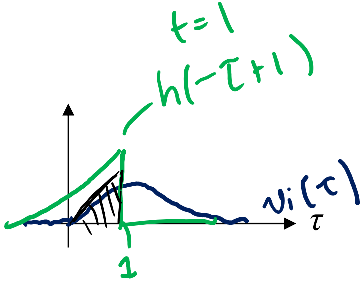
→ for causal systems $h(t)$ is zero for $t < \emptyset$ (real systems can't predict the future)

flip shift & integrate

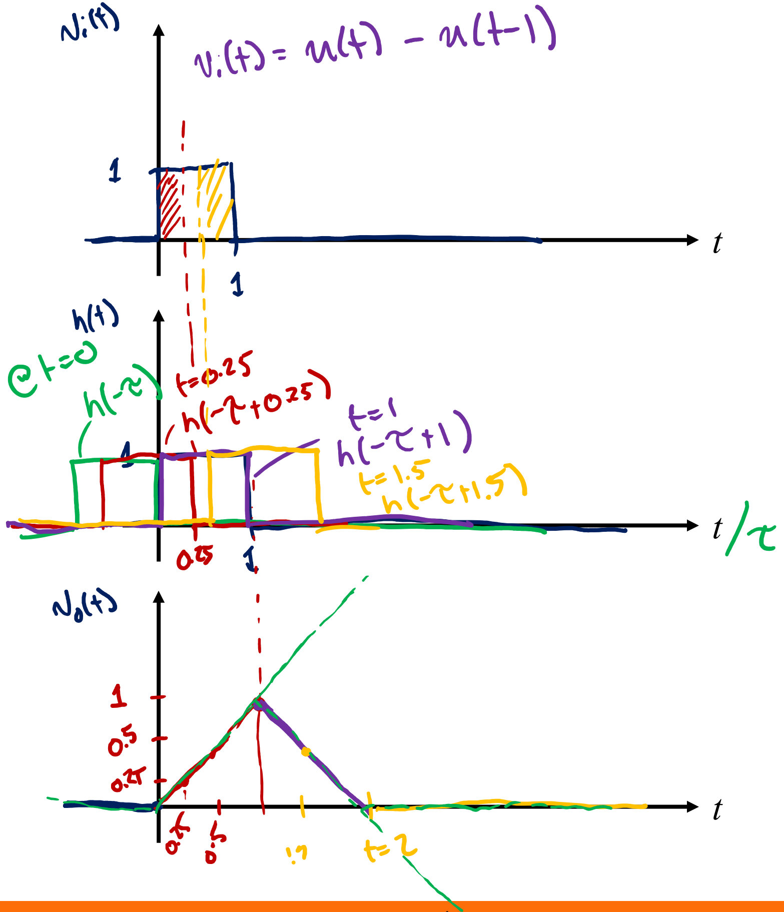
$$v_o(t) = \int_0^\infty h(-\tau + t)\, v_i(\tau)\, d\tau$$







Graphical Convolution



$$v_o(t) = \int_0^\infty h(t-\tau)\, v_i(\tau)\, d\tau$$

flip shift & integrate

$$V_I(s) = H(s) = \frac{1}{s} - \frac{1}{s}e^{-s}$$

$$V_o(s) = V_I(s)H(s) = \left(\frac{1}{s} - \frac{1}{s}e^{-s}\right)^2$$

$$= \frac{1}{s^2} - 2\frac{1}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}$$

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = r(t) - 2r(t-1) + r(t-2)$$

✓

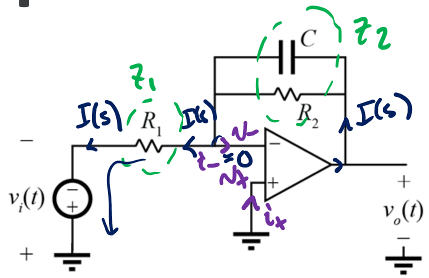
$$v_o(t) = r(t) - 2r(t-1) + r(t-2)$$

$$= t\,u(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$$

Example Problem

$z_1 = R_1$

$z_2 = \frac{1}{sc} \| R_2$



Ideal op-amp assumptions:
if there is negative feedback

(1) Virtual short : $v_+ = v_-$ $\rightarrow$ $V_+(s) = V_-(s)$

(2) $i_+ = i_- = \phi$ $\longrightarrow$ $I_+(s) = I_-(s) = \phi$

Inverting op-amp configuration $V_o(s) = \frac{-z_2}{z_1}\left(-V_I(s)\right)$

$I_-(s) = \phi$, $V_-(s) = \phi$

$$I(s) = \frac{0 - (-V_I(s))}{z_1} = \frac{V_I(s)}{z_1}$$

$$V_o(s) = V_-(s) + I(s)\,z_2 = \phi + \frac{V_I(s)}{z_1} z_2 \longrightarrow \boxed{V_o = \frac{z_2}{z_1} V_I(s)}$$

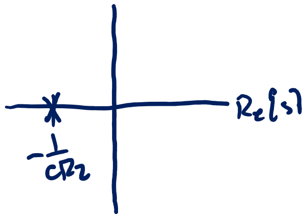
$$z_2 = \frac{\frac{1}{sc} R_2}{R_2 + \frac{1}{sc}} = \frac{\frac{1}{C}}{s + \frac{1}{CR_2}}$$

$z_1 = R_1$

$$V_o(s) = \underbrace{\frac{R_2}{R_1} \frac{\frac{1}{CR_2}}{s + \frac{1}{CR_2}}}_{H(s) = \text{Transfer function}} V_I(s)$$

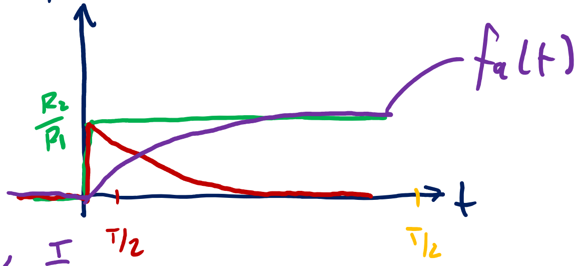
single pole @ $s = \frac{-1}{CR_2}$

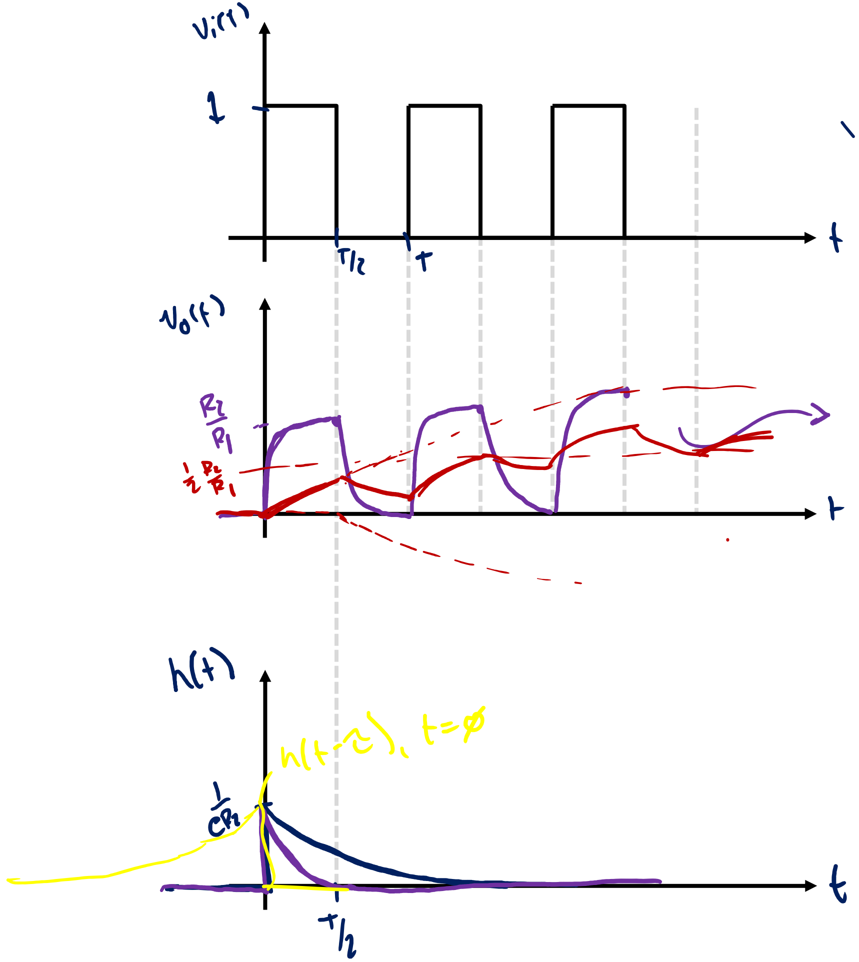
pole-zero plot



Poles in open LHP
BIBO stable

$$f_a(t) = \frac{R_2}{R_1}\left[1 - e^{-\frac{1}{R_2C}t}\right]u(t)$$





$\tau = R_2C \ll \frac{T}{2}$

$\tau = R_2C \gg \frac{T}{2}$

$$H(s) = \frac{R_2}{R_1}\,\frac{\frac{1}{CR_2}}{s + \frac{1}{CR_2}}$$

$$h(t) = \frac{1}{CR_1}\,e^{-\frac{1}{CR_2}t}\,u(t)$$

$$v_o(t) = \int_0^{\infty} h(t-\tau)\,v_i(\tau)\,d\tau$$