A lecture covering the bulk charge model and the charge sheet model used in the description of long-channel MOSFETs. These models are identified by their primary assumptions:

- **Bulk Charge Model**: The expansion of the depletion width from source to drain is modeled as an additional bulk charge appearing in the channel.

- **Charge Sheet Model**: The inversion layer is treated as a conductive sheet having a thickness of 0.

### 2 Quantitative MOSFET Modeling

#### 2.1 Bulk-Charge Model

The square-law model assumes that, above threshold, the charge in the channel is exclusively due to inversion charge. In this model, the threshold voltage accounts for the maximum amount of depletion charge induced in the semiconductor at the onset of inversion. This depletion charge was treated as a constant value from the source to the drain region. The depletion charge, strictly speaking, must increase slightly with increasing gate-voltage. This is true at all values of drain voltage and gives rise to an approximately uniform increase in depletion charge but is usually negligible in strong inversion due to a high density of inversion charge. For large drain voltages, there is an additional increase in depletion charge due to an increase in the reverse bias potential between the drain and channel regions. Thus, the depletion width will vary in the direction from source to drain, being higher near the drain.

\[
W(y) = \sqrt{\frac{2\epsilon_s(2\phi_F + \phi(y))}{eN_A^-}}
\]

\[
W(0) = \sqrt{\frac{2\epsilon_s(2\phi_F)}{eN_A^-}}
\]

\[
W(L) = \sqrt{\frac{2\epsilon_s(2\phi_F + V_D)}{eN_A^-}}
\]
This gives rise to an additional difference in bulk charge as a function of $y$ along the source to drain.

$$\Delta Q_{Dep}(y) = -eN_A^-(W(y) - W(0))$$

$$= -eN_A^W(0) \left( \frac{W(y)}{W(0)} - 1 \right)$$

$$= -eN_A^W(0) \left( \frac{\sqrt{1 + \frac{\phi(y)}{2\phi_F}} - 1}{2\phi_F} \right)$$

$$= -V_B C_{ox} \left( \sqrt{1 + \frac{\phi(y)}{2\phi_F}} - 1 \right)$$

...where $V_B$ is the voltage drop across the depletion charge at the onset of inversion.

$$V_B \equiv \frac{eN_A^-W(0)}{C_{ox}}$$

Thus, we add this correction to determine the new quantity of inversion charge available for conduction. Since we have a new charge component, the inversion charge will no longer simply be the negative of the charge placed at the gate. The gate charge is now shared between depletion charge and inversion charge. Thus, the available inversion charge for conduction will be lower than predictions from the square-law model.

$$Q_G = -Q_S = -(Q_{Inv}(y) + \Delta Q_{Dep}(y))$$

$$Q_{Inv}(y) = -Q_G - \Delta Q_{Dep}(y)$$

$$= -C_{ox} (V_G - V_T - \phi(y)) - \Delta Q_{Dep}(y)$$

$$= -C_{ox} (V_G - V_T - \phi(y)) + V_B C_{ox} \left( \sqrt{1 + \frac{\phi(y)}{2\phi_F}} - 1 \right)$$

$$= -C_{ox} \left( V_G - V_T - \phi(y) - V_B \left( \sqrt{1 + \frac{\phi(y)}{2\phi_F}} - 1 \right) \right)$$
This expression can be integrated to determine the total drain current.

\[ I_D = -\frac{Z}{L} \mu_{eff,n} \int_0^{V_D} Q_{inv}(y) d\phi \]

\[ = -\frac{Z}{L} \mu_{eff,n} \int_0^{V_D} -C_ox \left( V_G - V_T - \phi(y) - V_B \left( \sqrt{1 + \frac{\phi(y)}{2\phi_F}} - 1 \right) \right) d\phi \]

\[ = \frac{Z}{L} \mu_{eff,n} C_ox \int_0^{V_D} \left( V_G - V_T - \phi(y) - V_B \left( \sqrt{1 + \frac{\phi(y)}{2\phi_F}} - 1 \right) \right) d\phi \]

\[ = \frac{Z}{L} \mu_{eff,n} C_ox \left\{ (V_G - V_T) V_D - \frac{V_D^2}{2} - \frac{4}{3} V_B \phi_F \left( \left( 1 + \frac{V_D}{2\phi_F} \right)^{3/2} - \left( 1 + \frac{3V_D}{4\phi_F} \right) \right) \right\} \]

\[ \text{Square-Law} \]

\[ \text{Bulk-Charge Correction} \]

From this expression, it is evident that the bulk charge correction to the square-law formalism is always negative—that is, square-law is an over-estimate of the true current.

The square-law model always overestimates the true drain current

The magnitude of the correction scales with \( V_B \) and therefore the dopant density in the bulk. Therefore, as the dopant density is increased, the square-law model becomes increasingly less accurate.

2.2 Charge-Sheet Model

2.2.1 Problems with the Square-Law Model and Bulk Charge Model

Both the square-law model and the bulk charge model neglect subthreshold conduction. Below threshold, the inversion charge density is assumed to be zero. This predicts zero current flowing below threshold, which we know to be false.

Inversion charge is neglected below threshold for square law and bulk charge models.

Additionally, the square law models and bulk charge models do not self-saturate. One must impose an ad-hoc correction by restricting assuming the channel is pinched off above the saturation voltage \( V_{D_{sat}} \). The charge-sheet model removes these assumptions.

2.2.2 Charge-Sheet Formalism

Recall that the drain current can be expressed as follows (assuming that the current is due to drift):

\[ I_D = eZ \int_0^{x_c} n(x,y) \mu_n \frac{d\phi}{dy} dx \]

To derive the square-law equation, we assumed that \( \frac{d\phi}{dy} \) was constant with respect to \( x \). Here, we do not make that assumption. To move forward, similar to how we defined the effective mobility,
we can re-write the above using two new definitions:

\[ I_D = eZ \left( \int_0^{X_c} n(x,y) dx \right) \left( \frac{1}{\int_0^{X_c} n(x,y) dx} \right) \int_0^{X_c} n \mu_n \frac{d\phi}{dy} \ dx \]

\[ = eZ N(y) \left\langle \mu_n \frac{d\phi}{dy} \right\rangle \]

Where \( N(y) = \int_0^{x_c} n(x,y) \ dx \) and \( \left\langle \mu_n \frac{d\phi}{dy} \right\rangle = \frac{1}{N(y)} \int_0^{x_c} n \mu_n \frac{d\phi}{dy} \ dx \). If we treat the mobility appearing in the bracketed term as a number – the effective mobility–then it can be pulled out of the integral:

\[ \left\langle \mu_n \frac{d\phi}{dy} \right\rangle = \mu_{eff,n} \left\langle \frac{d\phi}{dy} \right\rangle = \mu_{eff,n} \frac{d\bar{\phi}}{dy} \]

Another way to justify this is to think of the simplification as a defining relation for the potential \( \bar{\phi} \):

\[ \frac{d\bar{\phi}}{dy} \equiv \frac{1}{\mu_{eff,n}} \left\langle \mu_n \frac{d\phi}{dy} \right\rangle \]

Thus, we can write the current as:

\[ I_D = eZ \mu_{eff,n} N(y) \frac{d\bar{\phi}}{dy} \]

To move forward, we need an expression for \( \frac{d\bar{\phi}}{dy} \) in terms of the electrostatic potential and the carrier concentration. Referring to the definition of the carrier concentration, and after applying the Boltzmann approximation, \( N \) can be shown to be equal to:

\[ N(y) = \int_0^{x_c} n(x,y) \ dx \]

\[ = \int_0^{x_c} n_i \exp \left( \frac{E_F - E_i(x)}{k_b T} \right) \ dx \]

\[ = \int_0^{x_c} n_i \exp \left( \frac{E_F - E_i(\infty) + E_i(\infty) - E_i(x)}{k_b T} \right) \ dx \]

\[ = \int_0^{x_c} n_i \exp \left( \frac{E_F - E_i(\infty)}{k_b T} \right) \exp \left( \frac{E_i(\infty) - E_i(x)}{k_b T} \right) \ dx \]

\[ = \int_0^{x_c} n_i \exp(-\beta \phi_F) \exp(\beta \phi) \ dx \]

\[ = \int_0^{x_c} n_i \exp(\beta (\phi - \phi_F)) \ dx \]

...where \( \beta \equiv \frac{e}{k_b T} \). Next, we expand the bulk potential near \( \phi_F \), taking the first term in the expansion:

\[ N(y) \approx \int_0^{x_c} n_i \exp(\beta (\phi - \phi_F)) \ dx \]
Differentiation of the above expression for \(N(y)\) yields the following:

\[
\frac{dN}{dy} = \beta \left( \frac{d\phi}{dy} - \frac{d\bar{\phi}_F}{dy} \right) N(y)
\]

After rearranging, we get the following...

\[
\frac{d\bar{\phi}_F}{dy} = \frac{d\phi}{dy} - \frac{1}{\beta} \frac{dN}{dy} = \frac{d\phi}{dy} - \frac{1}{\beta} \frac{d\ln N}{dy}
\]

...where \(\beta \equiv \frac{e}{k_B T}\). Replacing the \(\phi\) in the drain current expression with \(\bar{\phi}_F\), we can substitute the above expression into the expression for the drain current:

\[
I_D = eZ \mu_{eff,n} N(y) \left( \frac{d\phi}{dy} - \frac{1}{\beta} \frac{dN}{dy} \right)
\]

This equation can be rearranged in the following form:

\[
\frac{dN}{dy} - \beta \frac{d\phi}{dy} N(y) = -\frac{\beta}{Ze \mu_{eff,n}} I_D
\]

We recognize this as a first-order linear differential equation. The solution to this equation is given by:

\[
N(y) = -\frac{\beta}{Ze \mu_{eff,n}} I_D \exp \left( \beta \phi(y) \int_0^y \exp \left( -\beta \phi(y') \right) dy' + N(0) \exp \left( \beta \left( \phi(y) - \phi(0) \right) \right) \right)
\]

The essence of the Charge-Sheet model is that the inversion charge is assumed to be fixed at the oxide-silicon interface. Thus, we can replace \(\phi\) with \(\phi_s\), the surface potential.

\[
N(y) = -\frac{\beta}{Ze \mu_{eff,n}} I_D \exp \left( \beta \phi_s(y) \int_0^y \exp \left( -\beta \phi_s(y') \right) dy' + N(0) \exp \left( \beta \left( \phi_s(y) - \phi_s(0) \right) \right) \right)
\]

Coupling this with the usual Poisson’s equation in the oxide and the semiconductor:

\[
\nabla^2 \phi = 0 \quad \text{(oxide)}
\]

\[
\nabla^2 \phi = -\frac{e \left( p - N_A^- \right)}{\epsilon_s} \quad \text{(silicon)}
\]

The boundary condition at the oxide-semiconductor interface is given by Gauss’ law applied to the interface:

\[
D(0^+) - D(0^-) = Q_{IT}
\]

\[
\epsilon_{ox} \left( \frac{\partial \phi}{\partial x} \right)_{0^-} - \epsilon_s \left( \frac{\partial \phi}{\partial x} \right)_{0^+} = -eN(y) \quad \text{(Gauss’ law)}
\]

...where we acknowledge the inversion charge \(Q_{Inv} = -eN(y)\) as taking place of the interface charge, where \(0^+\) refers to the position along the x-axis at the semiconductor side of the oxide-
silicon interface and $0^-$ on the oxide side. Subject to the boundary condition, \( \phi(0, y) = \phi_s \), Poisson's equation can be solved in the silicon and the oxide and differentiated to evaluate the boundary condition regarding the displacement field used to compute \( N(y) \) self-consistently. Once \( N(y) \) is known, the integral equation for \( \phi_s \) can be evaluated. See handout – Brews, Solid-State Electronics, 21, 345-355, (1978) – for circumstantial simplifications.