CS311 — Induction Practice — Spring 2008

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The following simple arithmetic problems are for practice writing inductive proofs. All can be proved inductively, although some can also be proved noninductively. Most of the problems come from the *Introduction to Arithmetic* of Nicomachus of Gerasa (fl. c. 100 CE).

1. Show that the sum of the first n powers of 2 is $2^n - 1$; that is $2^n - 1 = \sum_{k=0}^{n-1} 2^k$.

2. Show that $r - 1$ times the sum of the first n powers of r is $r^n - 1$, i.e., $r^n - 1 = (r-1)\sum_{k=0}^{n-1} r^k$. For example, $2 \times (1 + 3 + 9 + 27) = 80 = 81 - 1 = 3^4 - 1$.

3. Show
$$\frac{1}{1} + \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^n} = \frac{\frac{r^n-1}{r-1} + r^n}{r^n}.$$

The following problems concern *figurate numbers*, that is, numbers in the shapes of triangles, squares, etc.

Definition: The *triangular numbers* are defined

$$\begin{aligned} \triangle_1 &= 1, \\ \triangle_n &= \triangle_{n-1} + n, \quad n > 1 \end{aligned}$$

Thus the triangles are $1, 3, 6, 10, \ldots$.

4. Show $\triangle_n = n(n+1)/2$.

5. Show
$$\frac{n+1}{n} = \frac{\triangle_{2n+1}}{\triangle_{2n}}, \quad n \geq 1.$$

6. Show
$$\sum_{k=2}^{n} \frac{1}{\triangle_k} = \frac{\triangle_{n-1}}{\triangle_n}, \quad n \geq 2.$$

Definition: The *square numbers* are defined $\square_n = n^2$. Thus the squares are $1, 4, 9, 16, \ldots$.

7. Show that $\square_n$ is the sum of the first n odd integers; for example, $3^2 = 1 + 3 + 5$.

8. Show that the nth square is the sum of the nth and $n-1$st triangles, $\square_n = \triangle_n + \triangle_{n-1}$. (For what values of n does this statment make sense?)

9. Show that one more than 8 times a triangle is a square, that is, $8\triangle_n + 1$ is square.

10. Show

$$\Box_n = 1 + 2 + 3 + \cdots + (n-1) + n + (n-1) + \cdots + 3 + 2 + 1.$$

For example, $3^2 = 9 = 1 + 2 + 3 + 2 + 1$.

Definition: The nth oblong number is $\mathtt{O}_n = n(n+1)$. Thus the oblong numbers are $2, 6, 12, 20, \ldots$.

11. Show that the nth oblong number is the sum of the first n even numbers. For example, $\mathtt{O}_3 = 12 = 2 + 4 + 6$.

12. Show that the $2n$th triangle is the sum of the nth square and the n oblong, $\triangle_{2n} = \Box_n + \mathtt{O}_n$.

13. List the square and oblong numbers in an alternating sequence,

$$\Box_1, \mathtt{O}_1, \Box_2, \mathtt{O}_2, \Box_3, \mathtt{O}_3, \ldots.$$

That is,

$$1, 2, 4, 6, 9, 12, 16, 20, \ldots.$$

Show that the sum of any two consecutive numbers in this sequence is a triangular number.

Definition: The *pentagonal numbers* are defined

$$\begin{aligned} \Pi_1 &= 1, \\ \Pi_n &= \Pi_{n-1} + 3n - 2, \quad n > 1 \end{aligned}$$

Thus the pentagons are $1, 5, 12, 22, \ldots$.

14. Show that the nth pentagonal number is the sum of the first n numbers in the series $1, 4, 7, 10, \ldots$ (increasing by 3s). For example, $\Pi_3 = 12 = 1 + 4 + 7$.

15. Show that the nth pentagonal number is the sum of the nth square and the $n-1$st triangle, i.e., $\Pi_n = \Box_n + \triangle_{n-1}$, for $n > 1$.

16. Show that 1 more than 24 times a pentagon is a square, that is, $24\Pi_n + 1$ is square. *Hint:* Write out the cases $n = 1, 2, 3, 4$ to get a formula for the square.

Definition: The nth *hexagonal number* H_n is equal to the sum of the first n numbers in the sequence $1, 5, 9, 13, \ldots$ (increasing by 4s). Thus the hexagons are $1, 6, 15, 28, \ldots$.

17. Show that the nth hexagon is the sum of the nth pentagon and the $n-1$st triangle, $\mathrm{H}_n = \Pi_n + \triangle_{n-1}$.

Definition: Let Φ_n^s represent the nth *figurate number* with s sides, so, for example, $\triangle_n = \Phi_n^3, \Box_n = \Phi_n^4$, etc. In general,

$$\begin{aligned} \Phi_1^s &= 1, \\ \Phi_{n+1}^s &= \Phi_n^s + (s-2)n + 1, \quad n \geq 1. \end{aligned}$$

(Actually, the definition of Φ_n^s can be extended to $n = 0$. What should be the value of Φ_0^s?)

18. Show $\Phi_n^{s+1} = \Phi_n^s + \triangle_{n-1}$, for $s \geq 2$.

19. Show $\Phi_n^s = n + (s-2)\triangle_{n-1}$ for $s \geq 2$.

20. Show that Φ_n^s is the sum of the first n numbers in the sequence

$$1, \quad (s-2)+1, \quad 2(s-2)+1, \quad 3(s-2)+1, \quad \ldots$$

(increasing by $s-2$).

21. List the odd numbers:

$$1, 3, 5, 7, 9, 11, 13, \ldots$$

The first is a cube, 1. The sum of the next two is a cube, $3 + 5 = 8 = 2^3$. The sum of the next three is a cube, $7 + 9 + 11 = 27 = 3^3$. And so forth. Express the general proposition mathematically and prove or disprove it.