COSC 317 Worksheet 1

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By working through this material you will gain experience with developing a mathematical theory. In the following, you will find definitions, examples, theorems, and problems. Usually I will give you the definitions, but sometimes I will ask you to complete them. This will give you experience in mathematical definition. Sometimes I provide examples of a defined concept, but you should show that the example fits the definition. Theorems are generally presented without proof, because it is your task to prove them. Problems, of course, should be solved by you. In summary, your assignment is to prove the theorems, solve the problems, check the examples, and (where asked) complete the definitions. This is the best way to learn mathematics: by doing it!

Generally speaking, you should work through these assignments in order, since they are organized to lead you along. However, if you get stuck on one item, it is OK to skip it and move on, but you should not do this too often or you will get lost. Please **don't** try to find this material online or in books, since if you do, you will not benefit from working it out yourself. Let's begin!

1. Posets

DEFINITION 1.1 (poset and partial order). A *poset* (*partially ordered set*) $(P, \sqsubseteq)$ is a set P together with a binary relation $\sqsubseteq$ that is a *partial order*, that is:

P1 (Reflexive): For all $x \in P$, $x \sqsubseteq x$.
P2 (Antisymmetric): For all $x, y \in P$, if $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$.
P3 (Transitive): For all $x, y, z \in P$, if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$.

We can pronounce $x \sqsubseteq y$ as "x is part of y," "x is less than or equal to y," "x is below or the same as y," "x is inferior or equal to y," etc. $x \sqsupseteq y$ means $y \sqsubseteq x$; we can pronounce it "contains," "is above or the same as," or "is superior or equal to." Obviously, if $\sqsubseteq$ is a partial order, then so is its *converse* $\sqsupseteq$. The *strict order* $x \sqsubset y$ is equivalent to $x \sqsubseteq y$ but $x \neq y$, and $x \sqsupset y$ is defined analogously. We say that x is a *proper* part of / less than / below / inferior to y.

DEFINITION 1.2 (strict partial order). A relation $\sqsubset$ on a set P is called a *strict partial order* if

S1 (Irreflexive): $x \sqsubset x$ does not hold for any $x \in P$.
S2 (Transitive): For all $x, y, z \in P$, if $x \sqsubset y$ and $y \sqsubset z$, then $x \sqsubset z$.

DEFINITION 1.3 (comparable). Elements x and y of a poset $(P, \sqsubseteq)$ are said to be *comparable* if either $x \sqsubseteq y$ or $y \sqsubseteq x$ (or both), and *incomparable* otherwise.

EXAMPLE 1.1. Let P be a set of people and define $x \sqsupset y$ to mean that x is an ancestor of y. Then $(P, \sqsubseteq)$ is a poset.

EXAMPLE 1.2. Let P be the set of statements in a program. Define $x \sqsubset y$ to mean that statement x depends on statement y (for example, x might use a variable to which y assigns). Then $(P, \sqsubseteq)$ is a poset.

EXAMPLE 1.3. If X is a set of integers, then $(X, \leq)$ is a poset.

EXAMPLE 1.4. If X is a set of real numbers, then $(X, \leq)$ is a poset.

EXAMPLE 1.5. If $X = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}\}$, then $(X, \subseteq)$ is a poset.

EXAMPLE 1.6. If X is any set of sets, then $(X, \subseteq)$ is a poset.

EXAMPLE 1.7. Let P be a set of integers and let $m|n$ mean that m evenly divides into n (i.e., with no remainder). Then $(P, |)$ is a poset.

EXAMPLE 1.8. Let $\mathbf{2} = \{0, 1\}$, representing the truth values *false* and *true*. Show that $(\mathbf{2}, \rightarrow)$, where "$\rightarrow$" is material implication, is a poset.

EXAMPLE 1.9. Let P be a set of strings and let $x \sqsubseteq y$ mean that x is a substring of y. Then $(P, \sqsubseteq)$ is a poset.

EXAMPLE 1.10. Let N be the nodes in a tree, and let $x \sqsubseteq y$ mean that node x is on a branch descending from y (or is y itself). Then $(N, \sqsubseteq)$ is a poset.

EXAMPLE 1.11. If V is the set of vertices (nodes) in an acyclic directed graph, define $x \sqsupseteq y$ to mean that there is a path from vertex x to vertex y. $(V, \sqsupseteq)$ is a poset. Why aren't cycles allowed?

Note that the previous two examples imply that any tree and any directed acyclic graph *induces* a partial order. These facts are useful, since it permits you to construct examples and counter-examples for theorems and conjectures.

THEOREM 1.1. A strict partial order is *asymmetric*; that is, if $x \sqsubset y$ then it is not the case that $y \sqsubset x$.

THEOREM 1.2. (A) If $\sqsubset$ is a strict partial order, and $x \sqsubseteq y \Leftrightarrow x \sqsubset y \vee x = y$, then $\sqsubseteq$ is a (nonstrict) partial order. (B) If $\sqsubseteq$ is a (nonstrict) partial order, and $x \sqsubset y \Leftrightarrow x \sqsubseteq y \wedge x \neq y$, then $\sqsubset$ is a strict partial order.

DEFINITION 1.4 (cover). In a poset, an element x *covers* an element y if $x \sqsupset y$ and there does not exist a z such that $x \sqsupset z \sqsupset y$. In effect, x is an "immediate superior" of y.

Problem 1.1. Describe the conditions under which an element of a poset does not have a cover.

Definition 1.5 (Hasse diagram). A *Hasse diagram* of a poset has a node for each element of the poset and a descending line from each element to those it covers. (This is another convenient mechanism for constructing examples and counter-examples.)

Problem 1.2. Draw a Hasse diagram for each of the preceding examples of posets (Exs. 1.1–1.11) or explain why it is impossible to do so. (Pick non-trivial examples for the sets.)

Theorem 1.3. There are exactly three partial orderings of a set of two elements. (Draw the Hasse diagram of each.)

Theorem 1.4. If $(P, \sqsubseteq)$ is a poset and $S \subseteq P$, then $(S, \sqsubseteq)$ is a poset.

Definition 1.6 (upper bound). For a poset $(P, \sqsubseteq)$, an element $x \in P$ is an *upper bound* of a subset $S \subseteq P$ if and only if $y \sqsubseteq x$ for all $y \in S$.

Definition 1.7 (lower bound). Analogous: you define it.

You will notice that a *principle of duality* applies to posets: since $\sqsubseteq$ is a p.o. if and only if $\sqsupseteq$ is a p.o., upper bounds correspond to lower bounds, etc. In effect, we can turn a Hasse diagram upside down and still have a Hasse diagram.

Definition 1.8. (self-dual) A poset is *self-dual* if its Hasse diagram looks the same when it is flipped vertically (ignoring node labels). This is an informal definition. How would you make it more formal?

Theorem 1.5. There are posets with subsets that do not have an upper bound. The same applies for lower bounds.

Problem 1.3. Which of the example posets (Exs. 1.1–1.11) are self-dual?

Definition 1.9 (least upper bound). x is a *least upper bound* of $S \subseteq P$ if x is an upper bound of S and $x \sqsubseteq y$ for every upper bound y of S.

Definition 1.10 (greatest lower bound). Analogous.

Problem 1.4. Let S be a nonempty subset of the poset described in Ex. 1.10. Must S have a lub? Describe it.

Theorem 1.6. If S has a least upper bound, then it is unique, and we can write $\operatorname{lub} S$ for it. Likewise, if S has a greatest lower bound, we write $\operatorname{glb} S$ for it.

Theorem 1.7. There are posets with subsets that have upper bounds, but not lubs. Likewise for lower bounds.

Definition 1.11 (maximal and minimal elements). If S is a subset of a poset P, then $m \in S$ is called a *maximal element* of S if there is no $x \in S$ with $x \sqsupset m$. *Minimal element* is defined analogously.

Problem 1.5. Describe the maximal and minimal elements of any nonempty subset of the poset described in Ex. 1.10.

Theorem 1.8. There are posets with subsets that have no maximal element. Likewise for minimal.

Theorem 1.9. There are posets with subsets that have more than one maximal element. Likewise for minimal.

Theorem 1.10. Any finite nonempty subset of a poset has minimal and maximal elements.

Definition 1.12 (greatest and least elements). An element $g \in S$ of a subset S of a poset is a *greatest element* if $g \sqsupseteq x$ for every $x \in S$. Likewise for *least element*.

Problem 1.6. Give an example of a poset with a greatest element and two minimal elements, but no least element.

Problem 1.7. Does the poset described in Ex. 1.10 have a greatest element? Least element?

Theorem 1.11. If a subset of a poset has a greatest element, then it is unique, and if it has a least element, it is unique.

Definition 1.13 (top and bottom elements). If the entire poset has a greatest element, then it is often written $\top$ and called the *top* element. Likewise, if the entire poset has a least element, it is called *bottom* and written $\bot$.