COSC 317 Worksheet 2

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2. Lattices

Definition 2.1 (meet and join). For $x, y \in P$, their *meet* is defined to be their greatest lower bound (if it exists): $x \wedge y = \text{glb}\{x, y\}$. Likewise, the *join* is defined $x \vee y = \text{lub}\{x, y\}$.

Problem 2.1. For the poset $(\mathbb{R}, \leq)$ describe the results of the meet and join operations.

Problem 2.2. Let P be a set of sets. What are the meet and join operations in the poset $(P, \subseteq)$?

Problem 2.3. Let P be a set of sets. What are the meet and join operations in the poset $(P, \supseteq)$? (Be careful! This means that $x \sqsubseteq y$ if and only if $x \supseteq y$.)

Problem 2.4. What are the meet and join operations in the poset of truth values $(\mathbf{2}, \Rightarrow)$?

Definition 2.2 (lattice). A poset P is called a *lattice* if for each $x, y \in P$, both $x \wedge y$ and $x \vee y$ exist.

Definition 2.3 (complete lattice). A lattice is *complete* if each of its subsets has both an lub and a glb.

Problem 2.5. Give an example of a lattice that is *not* complete.

Problem 2.6. Which of the example posets in Worksheet 1 are lattices? Which are complete?

Theorem 2.1. Any nonempty complete lattice has a greatest element $\top$ and a least element $\bot$.

Theorem 2.2. The dual of a lattice is a lattice; the dual of a complete lattice is a complete lattice.

Theorem 2.3. Let $\mathcal{P}(S)$ be the set of all subsets of S. Then $(\mathcal{P}(S), \subseteq)$ is a complete lattice. (What are its top and bottom elements?)

Whenever we define new operators, we should investigate immediately their properties. The meet and join operations satisfy a number of algebraic properties.

Theorem 2.4. In any poset, the meet and join operations, whenever they exist, satisfy the following algebraic laws:

L1 (Idempotent): $x \wedge x = x, \quad x \vee x = x.$
L2 (Commutative): $x \wedge y = y \wedge x, \quad x \vee y = y \vee x.$
L3 (Associative): $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad x \vee (y \vee z) = (x \vee y) \vee z.$
L4 (Absorption): $x \wedge (x \vee y) = x = x \vee (x \wedge y).$

THEOREM 2.5 (consistency). $x \sqsubseteq y$ if and only if $x \wedge y = x$ if and only if $x \vee y = y$.

THEOREM 2.6. If a poset P has a top element $\top$, then for all $x \in P$, $x \wedge \top = x$ and $x \vee \top = \top$. Similarly for $\bot$.

THEOREM 2.7 (isotone property). The meet and join operations in a lattice are *isotone*; that is, if $y \sqsubseteq z$, then $x \wedge y \sqsubseteq x \wedge z$ and $x \vee y \sqsubseteq x \vee z$.

THEOREM 2.8 (distributive inequalities). In any lattice,

$$\begin{aligned} x \wedge (y \vee z) &\sqsupseteq (x \wedge y) \vee (x \wedge z), \\ x \vee (y \wedge z) &\sqsubseteq (x \vee y) \wedge (x \vee z). \end{aligned}$$

PROBLEM 2.7. You might be surprised that these are inequalities and not equalities. Find a lattice for which equality does not apply.

THEOREM 2.9 (modular inequality). In a lattice, $x \sqsubseteq z$ implies $x \vee (y \wedge z) \sqsubseteq (x \vee y) \wedge z$.

THEOREM 2.10. In a lattice, $(a \vee b) \wedge (c \vee d) \sqsupseteq (a \wedge c) \vee (b \wedge d)$.

DEFINITION 2.4 (semilattice). A *semilattice* $(X, \diamond)$ is a set X and a binary operation $\diamond$ on X that is idempotent, commutative, and associative.

THEOREM 2.11. If P is a poset in which every pair of elements has a meet, then $(P, \wedge)$ is a semilattice. Likewise for $\vee$.

THEOREM 2.12. In a semilattice $(X, \diamond)$ define $x \sqsubseteq y$ to mean $x \diamond y = x$. Then $(X, \sqsubseteq)$ is a poset with $x \diamond y = \text{glb}\{x, y\}$.

THEOREM 2.13. A set with two binary operations obeying laws L1–L4 (Thm. 2.4) is a lattice, and conversely.

DEFINITION 2.5 (sublattice). If L is a lattice, then $S \subseteq L$ is a *sublattice* if every pair of elements of S has both a meet and a join in S (i.e., using the same meet and join as L).

THEOREM 2.14. Both the empty set and the singleton sets are sublattices of a lattice. (Always check "degenerate" cases such as these.)

PROBLEM 2.8. Give examples of (non-degenerate) sublattices of the example lattices from Worksheet 1.

THEOREM 2.15. If L is a complete lattice and $S \subseteq L$, and if (1) $\top \in S$ and (2) $\mathrm{glb}\, R \in S$ for every $R \subseteq S$, then S is a complete lattice.

PROBLEM 2.9. Give counter-examples showing that each of the two conditions in the preceding theorem are required.

DEFINITION 2.6 (direct product of posets). If P, Q are posets, their *direct product* $P \times Q$ is defined $(x, y) \sqsubseteq (x', y')$ if and only if $x \sqsubseteq x'$ in P and $y \sqsubseteq y'$ in Q.

THEOREM 2.16. The direct product of two lattices is a lattice.