Foundations of Field Computation

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Chapter 4

Basic Complex Analysis

Although real numbers are sufficient for most applications of field computation, complex numbers are sometimes required, as in Fourier analysis and the application of field computation is in quantum computation. Therefore the goal of this chapter is to provide an intuitive understanding of basic complex analysis, especially as it applies in Hilbert spaces; a systematic presentation of complex analysis is beyond its scope. In addition to standard material, this chapter includes a brief discussion of hyperbolic trigonometry and its applications in special relativity theory, which is intended to build intuition by stressing the analogies with ordinary (circular) trigonometry.

4.1 Argand diagram

As everyone knows, complex numbers involve $i = \sqrt{-1}$. However, it will be better at this point to forget about $\sqrt{-1}$ and understand complex numbers by means of the *Argand diagram* (Fig. 4.1). As a matter of history, mathematicians were dubious about imaginary numbers, and questioned their legitimacy, until familiarity with the Argand diagram showed that they could be thought of as ordinary two-dimensional vectors. For in the Argand diagram we simply represent the complex number $x + iy$ as a vector (x, y). (In this sense "i" can be thought of as a place holder or tag to distinguish the Y-coordinate from the X-coordinate.) Then operations on complex numbers can be interpreted as operations on two-dimensional vectors, without concern for $\sqrt{-1}$. When complex numbers are represented in this way, they are said to lie in the *complex plane*. Real numbers lie along the positive and

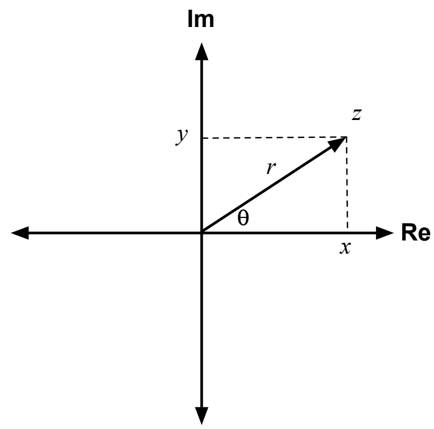

Figure 4.1: Argand diagram. $z = x + iy = re^{i\theta}$.

negative X-axis, and (pure) imaginary numbers along the positive and negative Y-axis; other points represent complex numbers with both (nonzero) real and imaginary parts. Therefore, in the complex plan the X-axis is called the *real axis* and the Y-axis is called the *imaginary axis*. (Why we should bother with complex numbers, and not simply make do with two-dimensional vectors, will become apparent as we proceed.)

Remark 4.1.1 *Notice that, unlike the real numbers, there is no natural sense in which the complex numbers can be ordered.*

Definition 4.1.1 (Cartesian components) *The* $\Re : \mathbb{C} \to \mathbb{R}$ *and* $\Im : \mathbb{C} \to \mathbb{R}$ *operators extract the Cartesian components (real and imaginary parts, respectively) of a complex number:*

$$\begin{aligned} \Re(x+iy) &= x, \\ \Im(x+iy) &= y. \end{aligned}$$

4.2 Geometrical Interpretations

The simplest use of the Argand diagram is to understand the addition and subtraction of complex numbers.

Definition 4.2.1 (Complex addition) *Addition (or subtraction) of complex numbers is equivalent to vector addition in the Argand diagram:*

$$(x+iy)+(x'+iy')=(x+x')+i(y+y').$$

Definition 4.2.2 (Complex multiplication) $(x+iy)(x'+iy')=(xx'-yy')+i(xy'+yx')$.

Remark 4.2.1 *The definition of multiplication may seem mysterious, but it is motivated by the equation $i^2=-1$. Thus,*

$$(x+iy)(x'+iy')=xx'+iyx'+iy'x+i^2yy'=(xx'-yy')+i(xy'+yx').$$

Further, we will see that it has important implications independent of $\sqrt{-1}$.

Definition 4.2.3 (Complex conjugate) *The complex conjugate $\overline{z}$ of a complex number z is obtained by negating its imaginary part:*

$$\overline{x+iy}=x-iy.$$

The notation z^* is also used for the complex conjugate.

Remark 4.2.2 *The complex conjugate reflects the vector across the real (X) axis. Symmetry suggests that there ought to be an operation to reflect a complex number $x+iy$ across the imaginary (Y), yielding $-x+iy$, but it is not especially useful, so it doesn't have a name. Of course, simple negation reflects a complex number across both axes simultaneously, $-(x+iy)=-x-iy$.*

Exercise 4.2.1 *Prove the following:*

$$\begin{aligned}\overline{\overline{x}} &= (x^*)^*=x\\ \overline{x+y} &= \overline{x}+\overline{y}\\ \overline{xy} &= \overline{x}\,\overline{y}\\ \overline{x/y} &= \overline{x}/\overline{y}\end{aligned}$$

Exercise 4.2.2 *Show that*

$$\Re z=\frac{z+\overline{z}}{2}\quad \text{and}\quad \Im z=\frac{z-\overline{z}}{2i}.$$

The Argand diagram suggests that the *magnitude* ("length") of a complex number is a significant quantity. The length, for course, is $\sqrt{x^2 + y^2}$, but this can be expressed conveniently in terms of the complex conjugate (which is one of the reasons the complex conjugate is useful), since:

$$(x + iy)(x - iy) = x^2 + ixy - ixy - i^2y^2 = x^2 + y^2.$$

Therefore we have:

Definition 4.2.4 (Magnitude) $|z| = \sqrt{z\overline{z}}$.

Remark 4.2.3 *Notice that this is consistent with the usual definition of the absolute value of a real number, since for a real* r, $\overline{r} = r$*; hence* $|r| = \sqrt{r\overline{r}} = \sqrt{r^2}$*. (Recall that, by convention,* $\sqrt{}$ *represents the nonnegative square root.)*

Remark 4.2.4 *The complex magnitude is a norm.*

Remark 4.2.5 *The distance between complex numbers is* $|z - z'|$*; it is the norm metric.*

Exercise 4.2.3 *What would be wrong with defining the magnitude of a complex number by* $|z| = \sqrt{z^2}$*. Would it be a norm? Would* $|z - z'|$ *be a metric?*

Exercise 4.2.4 *Show that* $|\Re z| \leq |z|$ *and* $|\Im z| \leq |z|$.

Exercise 4.2.5 *Show that* $|z - w| \geq ||z| - |w||$.

Proposition 4.2.1 *The reciprocal of a complex number is given by*

$$(x + iy)^{-1} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}. \tag{4.1}$$

Exercise 4.2.6 *Derive the preceding formula by solving* $wz = 1$ *for* w*; note that the real and complex parts of the equation can be solved separately.*

Exercise 4.2.7 *Write* z^{-1} *in terms of the complex conjugate and the magnitude. Does this simplify deriving Eq. 4.1?*

Exercise 4.2.8 *Derive the formula for dividing two complex numbers; you should get a formula in the form* $X + iY$.

Remark 4.2.6 *The complex numbers form an (algebraic)* field*; that is, there are both additive and multiplicative identities and inverses, both operations are communtative and associative, and multiplication distributes over addition.*

4.3 Polar representation

4.3.1 Defined

Of course two-dimensional vectors can be represented in polar coordinates as well as in rectangular coordinates, but the polar representation is especially relevant to complex numbers. The radius is, of course, the magnitude of the complex number, also called the *modulus*.

Definition 4.3.1 (Magnitude or modulus) $\mathrm{mod}\, z = |z| = \sqrt{x^2 + y^2}$.

The angle is measured counterclockwise from the positive X-axis and is called the *argument*, *phase*, *amplitude* or *angle* of the complex number. It can be defined as follows:

Definition 4.3.2 (Argument or phase) $\arg(x + iy) = \arctan(y/x)$.

Remark 4.3.1 *The mathematically most convenient way to measure angles is in* radians*, which is defined to be the area within a circle enclosed by twice the angle divided by the square radius of the circle. That is, if A is the area enclosed by the angle, then its radian measure is $\theta = 2A/r^2$. Since the circle has area πr^2, an angle of π radians corresponds to $180°$ (since twice the angle includes the whole area), $\pi/2$ radians corresponds to $90°$, 2π radians to $360°$, etc. In general, if α is an angle in degrees, then $\theta = 2\pi(\alpha/360°)$.*

Notation 4.3.1 *Because we are often interested in angles that are fractions or multiples of a complete cycle (2π radians), I have invented a kind of monogram, 2π, that I will use for 2π whenever it represents a complete cycle ($360°$).*[1]

Remark 4.3.2 *We write $\arctan(y/x)$ so that the signs of x and y can be used to determine the quadrant of the complex plane in which the number falls. Thus $\arctan(+1/+1) = \pi/4$, $\arctan(+1/-1) = 3\pi/4$, $\arctan(-1/-1) = 5\pi/4$, and $\arctan(-1/+1) = 7\pi/4$, even through they all represent only two slopes, $+1$ and -1.*

[1] It turns out that the convention of using a single symbol for 2π goes back at least as far as H. Laurent's *Traité D'Algebra* (1889). In recent years some mathematicians have proposed using τ (standing for one *turn*) for 2π, and others have advocated for a different monogram: π. See Palais (2001) and `http://www.math.utah.edu/~palais/pi.html` (accessed 2012-05-10).

Remark 4.3.3 *It is often useful to consider* arg *a* mutiple-valued function *(like* arcsin*,* arccos*, etc.). Thus, for example,*

$$\arg(-1) = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$$

Then, it is necessary to be clear about the range of arg*'s values; unless otherwise stated we will take it to be* $[0, 2\pi)$*. When the range is not important, we may equations such as*

$$\arg z = \theta \pmod{2\pi}$$

to indicate that angles are to be compared modulo 2π*.*

Remark 4.3.4 *If* z *is a complex number with magnitude* r *and phase* θ*, it's easy to see that the real part is given by* $\Re z = r\cos\theta$ *and the imaginary part by* $\Im z = r\sin\theta$*.*

Remark 4.3.5 *Notice every complex number has multiple polar representations (a property, of course, of any polar representation), since* $\sin\theta = \sin(2\pi+\theta)$ *and* $\cos\theta = \cos(2\pi+\theta)$*. In general, for any* $n = 0, \pm 1, \pm 2, \ldots$*,* $\sin\theta = \sin(2\pi n+\theta)$ *and* $\cos\theta = \cos(2\pi n+\theta)$*. We will see that this periodicity in the phase of complex numbers makes them especially convenient for representing periodic phenomena such as waves.*

Exercise 4.3.1 *Given* $z = re^{i\theta}$*, show geometrically that*

$$r\cos\theta = \frac{z+\overline{z}}{2}, \qquad r\sin\theta = \frac{z-\overline{z}}{2i}.$$

4.3.2 cis function

Suppose $r = |z|$ and $\theta = \arg z$; then it's easy to see:

$$z = \Re z + i\Im z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

This equation shows, in effect, how the complex number can be reconstituted from it magnitude and argument. For this purpose the "cis" ($\cos + i\sin$, pronounced "sis") function is often used.

Definition 4.3.3 (cis function) $\operatorname{cis}\theta = \cos\theta + i\sin\theta$.

Thus $z = r \operatorname{cis}\theta$, or, more generally,

$$z = |z| \operatorname{cis}(\arg z). \tag{4.2}$$

Proposition 4.3.1 $\operatorname{cis}\theta \ \operatorname{cis}\phi = \operatorname{cis}(\theta + \phi)$.

Exercise 4.3.2 *Prove this. Hint: recall from trigonometry,*

$$\begin{aligned} \sin(\theta + \phi) &= \sin\theta \ \cos\phi + \cos\theta \ \sin\phi, \\ \cos(\theta + \phi) &= \cos\theta \ \cos\phi + \sin\theta \ \sin\phi. \end{aligned}$$

Remark 4.3.6 *The equation* $\operatorname{cis}\theta \ \operatorname{cis}\phi = \operatorname{cis}(\theta + \phi)$ *suggests that* cis *has some similarities to the exponential function; we shall see that this is more than coincidental.*

4.4 Complex exponentials

4.4.1 Euler's Formula

4.4.1.1 Imaginary Exponentials

For the most part complex numbers have the same properties as real numbers, but of course it's necessary to analyse each property individually; here we will assume complex numbers are like real numbers unless stated otherwise. However, it is informative to look (informally) at the effect of taking the exponential of an imaginary number, $\exp(i\theta) = e^{i\theta}$. To do this we use the familiar power series for e^x:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Substituting $i\theta$ for x we have:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right), \end{aligned}$$

where in the last line the real terms have been separated from the imaginary terms. As it turns out, the first parenthesized formula is the series for $\cos\theta$ and the second is that for $\sin\theta$. Thus we discover,

Proposition 4.4.1 (Euler's Formula) $e^{i\theta} = \cos\theta + i\sin\theta = \operatorname{cis}\theta$.

It's now easy to discover the exponential of an arbitrary complex number $x + iy$:

$$e^{x+iy} = e^x e^{iy} = e^x \operatorname{cis} y.$$

There is, however, an even more fruitful way to look at the complex exponential, since from Eq. 4.2 we see that any complex number can be written as a complex exponential:

$$z = |z|e^{i \arg z}.$$

Or, looked at another way, $re^{i\theta}$ is a complex number with magnitude (radius) r and phase angle θ.

Exercise 4.4.1 *Show that* $|e^{x+iy}| = |e^x|$.

Exercise 4.4.2 *Use Euler's formula to prove that the following formulas are correct for real* θ*:*

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{4.3}$$

They also hold (by definition) for complex numbers.

Notice that -1 has a magnitude of 1 and a phase angle of π (i.e. 180°); thus we have the famous equation,

$$e^{\pi i} = -1.$$

Since a 2π (360°) rotation brings us back where we started, we also have the less famous formula,

$$e^{2\pi i} = 1$$

Obviously a phase angle that is any integral multiple of 2π will bring us back to 1. Similarly,

$$e^{\pi i/2} = i,$$
$$e^{3\pi i/2} = -i.$$

Exercise 4.4.3 *Show* $re^{i\theta} = re^{i(2\pi m+\theta)}$ *for* $m = 0, 1, \ldots$.

Exercise 4.4.4 *Show* $(re^{i\theta})^* = re^{-i\theta}$.

Exercise 4.4.5 *Show* $re^{i\theta} = re^{-i(2\pi-\theta)}$.

4.4.1.2 Multiplication

Proposition 4.4.2 *Complex numbers can be multiplied by multiplying their magnitudes and adding their phase angles:*

$$(re^{i\theta})(se^{i\phi}) = (rs)e^{i(\theta+\phi)}.$$

Remark 4.4.1 *This proposition provides a geometric interpretation of complex multiplication, based on the proportion,*

$$1 : z \; :: \; w : wz.$$

To multiply geometrically, construct a triangle with the sides 1 and z. Then construct a similar triangle on w, with the 1 of the first triangle corresponding to the w of the second. The result wz will be the side of the second triangle corresponding to the side z of the first. (Interestingly, this is exactly analogous to the construction Descartes used for defining the product of two real magnitudes; see ch. 4 of my book in progress, Word and Flux.)

Exercise 4.4.6 *Do the construction suggested by the preceding remark, and show that the magnitudes and phase angles are correct, as given in the proposition.*

Remark 4.4.2 *Therefore a complex number, as a vector, can be rotated by multiplying by a suitable imaginary power of e. That is, to rotate z counterclockwise through an angle of θ, use $e^{i\theta}z$; for a clockwise rotation use $e^{-i\theta}z$.*

Exercise 4.4.7 *Give a rule for dividing complex numbers in terms of their magnitudes and phase angles.*

Remark 4.4.3 *We have seen that we can consider the complex number $x+iy$ as a two dimensional vector (x, y) with ordinary vector addition and a special multiplication rule. Similarly, we can consider the complex number $re^{i\theta}$ as a pair (r, θ) with a special operation that multiplies the magnitudes and adds the phase angles. In particular,* whenever you have pairs of numbers for which you want to add the first components and multiply the second components, it may be worthwhile to think of them as complex numbers in polar coordinates. *We will see an example shortly (Section 4.4.2).*

4.4.1.3 Powers and Roots

Proposition 4.4.3 (De Moivre's Theorem) *A complex number can be raised to the $p \geq 1$ power by raising its magnitude to the p power and by multiplying its phase by p: $(re^{i\theta})^p = (r^p)e^{ip\theta}$.*

Roots can be extracted in a similar way, but complex number bring some additional complications, as we will see by considering the "n n-th roots of unity." Consider first the square-root; we want to consider complex numbers z satisfying $z^2 = 1$. Writing the equation in polar form, we have

$$1 = (re^{i\theta})^2 = r^2 e^{2i\theta}.$$

To solve this, we must have $r = 1$, but we may have θ be any angle such that $2\theta = 2\pi m$ (for some $m = 0, 1, \ldots$). If we restrict our attention to θ in the range $[0, 2\pi)$ (the *principal* square-roots), we see that $\theta = 0, \pi$ both solve the equation. Therefore, 1 has two square roots, $e^{0i} = 1$ and $e^{\pi i} = -1$. This is obvious enough, since $1^2 = (-1)^2 = 1$.

Now however we will apply the same method to determine the cube-roots of unity. Since $(re^{i\theta})^3 = r^3 e^{i3\theta}$, we again have $r = 1$, but now seek $\theta \in [0, 2\pi)$ such that $3\theta = 2\pi m$. Hence, $\theta = 0, (1/3)2\pi$ and $(2/3)2\pi$ are solutions.

Exercise 4.4.8 *Confirm that these θ are solutions.*

Hence, we find that 1 has three cube-roots, two of which are complex:

$$1, e^{2\pi i/3}, e^{2\pi i 2/3}.$$

In general we can see that 1 has n (principal) n-th roots, having phase angles satisfying $n\theta = 2\pi m$ ($m = 0, 1, \ldots$), so $\theta = 2\pi m/n$. Hence,

Proposition 4.4.4 *The n principal n-th roots of unity are:*

$$1, e^{i2\pi/n}, e^{i2\pi 2/n}, \ldots, e^{i2\pi(n-1)/n}.$$

In general, the principal values are $e^{i2\pi m/n}$, $m = 0, 1, \ldots, n-1$.

Proposition 4.4.5 *The n principal n-th roots of a complex number $z = re^{i\theta}$ are:*

$$\sqrt[n]{r},\ \sqrt[n]{r}\ e^{i(\theta+2\pi)/n},\ \sqrt[n]{r}\ e^{i(\theta+2\pi 2)/n}, \ldots, \sqrt[n]{r}\ e^{i[\theta+2\pi(n-1)]/n}.$$

In general, the principal values are

$$\sqrt[n]{r}\; e^{i(\theta+2\pi m)/n} = \sqrt[n]{r}\;\operatorname{cis}\left(\frac{\theta}{n} + 2\pi\frac{m}{n}\right)$$

for $m = 0, 1, \ldots, n-1$.

Exercise 4.4.9 *Prove this proposition.*

4.4.2 Periodic Change

4.4.2.1 Introduction to Periodic Change

The polar representation of complex numbers makes them especially convenient for representing periodic processes, especially those involving sinusoidal change.

Remark 4.4.4 *It is generally convenient to measure the rate of periodic change in radians per second, its* angular velocity *or* angular frequency*. This is generally symbolized by* ω*, so we may write* $\sin \omega t$*. In some cases it is more meaningful to measure the rate of periodic change by its* frequency*, measured in cycles per second or Hertz. This is generally symbolized by* f *or* ν*; since there are* 2π *radians per cycle,* $\sin \omega t = \sin 2\pi f t = \sin 2\pi\nu t$*, for example.*

Suppose we have an object rotating counter-clockwise at ω radians per second; then its motion in the plane of rotation can be written $z(t) = e^{i\omega t}$. This assumes that at time $t = 0$ the object is at an angle of zero, that is, at location $(1, 0)$, since $z(0) = e^{i\omega 0} = 1$. If instead it starts at the angle ϕ we simply write

$$z(t) = e^{i(\omega t+\phi)} = e^{i\phi}e^{i\omega t}.$$

The factor $e^{i\phi}$ simply advances the phase of the rotation by ϕ radians. Obviously, arbitrary phase shifts correspond to different imaginary exponentials.

A complex exponential representation of a periodic change may be advantageous even is it is not a circular motion in two dimensions. In some cases, two different aspects of the change correspond to the real and imaginary parts of a complex number. For example, in simple harmonic motion (such as a pendulum or oscillating spring), the position of the object is proportional to $\sin \omega t$ and its velocity is proportional to $\cos \omega t$. Therefore, the position and velocity can be combined into one complex number and written

$$\cos \omega t + i \sin \omega t = \operatorname{cis} \omega t = e^{i\omega t}.$$

(I've assumed that the position and velocity are measured in suitable units so that the motion in the complex plane is circular.)

Since the state of a simple harmonic oscillator is determined entirely by its displacement and velocity, the complex number $e^{i\omega t}$ corresponds to its state. In this case the complex plane is the (Poincaré) *phase space* of the oscillator, since it represents all its possible states. The curve $e^{i\theta} = 1$, that is, the unit circle, is this system's *trajectory* or *orbit* in phase space; it shows the possible sequence of states independent of time.

The complex exponential representation can yield additional insight into the structure of a periodic process. For example, in Newtonian mechanics the kinetic energy of a motion is proportional to the velocity squared, $K \propto \cos^2 \omega t$ in this case. With a suitable choice of units we can write $K = \cos^2 \omega t$. Also, in many simple harmonic systems the restoring force is proportional to the displacement ($F \propto -\sin \omega t$), so the potential energy, which is the integral of the force, is proportional to the square of the displacement, $U \propto \sin^2 \omega t$; with suitable units, $U = \sin^2 \omega t$. Hence the total energy in the system is

$$E = K + U = \cos^2 \omega t + \sin^2 \omega t = 1.$$

That is the total energy is conserved; the $\cos^2$ and $\sin^2$ terms reflect the fraction of the energy in the kinetic or potential form, respectively. That is, $K = E\cos^2 \omega t$ and $U = E\sin^2 \omega t$. Further, as $e^{i\omega t}$ rotates, we can see the energy shift back and forth between kinetic energy (proportional to the square of the real component, representing velocity) and potential energy (proportional to the square of the imaginary component, representing displacement).

Remark 4.4.5 *For the record, $K = mv^2/2$ and $U = kx^2/2$, where k is the force constant of an ideal spring, $F = -kx$ (Hookes Law). In this example, $v(t) = \omega \cos \omega t$ and $x(t) = \sin \omega t$, where the angular frequency is determined by $\omega^2 = k/m$.*

Even when there aren't two components corresponding to the real and imaginary parts, it may be advantageous to treat a sinusoidal motion as the real (or imaginary) part of a complex exponential, since it is often easier to manipulate exponentials than sines and cosines. That is, it may be convenient to treat a real signal $\cos \omega t$ as $\Re e^{i\omega t}$. As you probably know, periodic signals, such as sounds, can be broken down into sines and cosines (or into sines with phase shifts). Therefore, they can equally, and often more conveniently, be broken down into complex exponentials.

Finally, as you probably know, Fourier analysis involves breaking a periodic wave into sines and cosines, or into sinusoids at various phases. Therefore, it is not surprising that it can also be viewed as an analysis of a signal into complex exponentials with complex coefficients. We will take up these topics in Ch. 6, Fourier Analysis; here we mention them only to motivate the study of complex exponentials.

Exercise 4.4.10 *Let $z = \rho e^{i\theta}$. Show that*

$$ze^{i\omega t} + \overline{z}e^{-i\omega t} = 2\rho\cos(\theta + \omega t) = 2\rho\sin(\theta + \omega t + 2\pi/4).$$

Hint: Write $e^{\pm i\omega t}$ in cis *form. This shows that a "conjugate pair of complex exponentials" is equivalent to an "amplitude and phase-shifted sinusoid."*

Exercise 4.4.11 *Show that*

$$a\cos\omega t + b\sin\omega t = ze^{i\omega t} + \overline{z}e^{-i\omega t}, \quad \textit{where } z = \frac{a - ib}{2}.$$

Hint: Write the sine and cosine in their complex exponential forms. Thus a mixture of a sine and a cosine is equivalent to a conjugate pair of complex exponentials, which the preceding exercise shows to be equivalent to an amplitude and phase-shifted sinusoid.

Remark 4.4.6 *These two exercises show the equivalence of: (1) a mixture of a sine and cosine of the same frequency (with parameters a and b), (2) an amplitude and phase-shifted sinusoid (with parameters ρ and θ), and (3) a conjugate pair of complex exponentials (with parameters $\Re z$ and $\Im z$). Therefore, a Fourier series for a signal can be equivalently viewed as a superposition of: (1) in-phase sines and cosines, (2) sinusoids of the same kind but differing phases, or (3) conjugate pairs of complex exponentials.*

4.4.2.2 Phasors

In this section I will discuss briefly a technique used in electrical engineering for analyzing circuits; it also has applications to understanding signal processing in the dendritic trees of neurons. Many passive electrical components, such as resistors, capacitors and inductors (coils) are linear. So also, to a first approximation, the passive conductance and membrane capacitance of dendrites is linear. Linearity means that if we know the behavior of a system L on complex exponentials (i.e. sines and cosines) of various frequencies

$L(e^{i\omega_k t})$, then we know its behavior on any periodic signal $s(t) = \sum_k c_k e^{i\omega t}$. This is because,

$$L[s(t)] = L\left(\sum_k c_k e^{i\omega_k t}\right) = \sum_k c_k L(e^{i\omega_k t}).$$

It turns out that resistors, capacitors and inductors have only two effects on sine waves: to attenuate them and to shift their phase; so also RLC (resistor-inductor-capacitor) circuits have only these two effects. Therefore, the effects of these circuits and their components are conveniently represented by complex numbers $Z = Ae^{i\theta}$, where A represents an amplitude change and θ represents a phase shift.

Remark 4.4.7 *The impedance of a R-ohm resistor is R; that is, it does not affect the phase.*

Remark 4.4.8 *At a frequency of ω rad./sec., the impedance of a L-henry inductor is $i\omega L$.*

Remark 4.4.9 *At a frequency of ω rad./sec., the impedance of a C-farad capacitor is $1/i\omega C$.*

Exercise 4.4.12 *Write the impedance $1/i\omega C$ in rectangular form, that is, in the form $R + iX$.*

Remark 4.4.10 *When an impedance $Ae^{i\theta}$ is written in rectangular coordinates $R+iX$, the real part R is called a resistance and the imaginary part X is called a reactance. Therefore, any arbitrary RLC circuit, no matter how complicated has the effect of a resistance combined with a reactance. If the reactance is positive, it is called an* inductive *reactance; if it is negative, it is called a* capacitive *reactance. That is, an arbitrary RLC circuit behaves like a resistor combined with either an inductor (which causes phase leading) or a capacitor (which causes phase lagging)*

Electrical engineers often use the *phasor* notation $A\angle\theta$, read "A angle θ" for $Ae^{2\pi i\theta/360°}$. The notation may be used for a circuit that causes an amplitude change A and a phase shift of θ degrees, or for a periodic signal (at a certain frequency) of amplitude A and phase θ°. The notation is convenient because of the simple operation rules:

$$A\angle\theta \times B\angle\phi = AB\angle(\theta + \phi),$$
$$A\angle\theta / B\angle\phi = (A/B)\angle(\theta - \phi).$$

With this notation, voltage, current and impedance (voltage divided by current) can all be treated as phasor quantities.

4.4.2.3 Differential Equations

In section 4.4.1.1 we saw the relation between the MacLauren series for the exponential, sine and cosine functions; here we look at the relation between these functions from another perspective. These functions can also be defined in terms of simple differential equations. For example $f(x) = \sin x$ is the unique solution of $f''(x) = -f(x)$ with initial conditions $f(0) = 0$ and $f'(0) = 1$. Likewise, $f(x) = \cos x$ is the unique solution of the same equation but with initial conditions $f(0) = 1$, $f'(0) = 0$.

Exercise 4.4.13 *Show that $f(t) = \sin \omega t$ is a solution to $f''(t) = -\omega^2 f(x)$ with initial conditions $f(0) = 0$ and $f'(0) = \omega$. (You are not asked to prove uniqueness.)*

Exercise 4.4.14 *Show that $f(t) = \cos \omega t$ is a solution to $f''(t) = -\omega^2 f(x)$ with initial conditions $f(0) = 1$ and $f'(0) = 0$. (You are not asked to prove uniqueness.)*

These differential equations give us an alternate way of deriving Euler's formula from reasonable expectations about the meaning of $e^{i\theta}$. To see this, write

$$e^{i\theta} = E(\theta) + iF(\theta); \tag{4.4}$$

we will solve for E and F. Since $e^0 = 1$ we must have $E(0) = 1$ and $F(0) = 0$. Now differentiate Eq. 4.4 (assuming, or postulating, it differentiates normally), to get

$$ie^{i\theta} = E'(\theta) + iF'(\theta).$$

Substitute $\theta = 0$ and we discover (Show in detail!) that $E'(0) = 0$ and $F'(0) = 1$. Differentiating a second time yields

$$-e^{i\theta} = E''(\theta) + iF''(\theta).$$

Combining this and Eq. 4.4 shows

$$E''(\theta) + iF''(\theta) = -E(\theta) - iF(\theta).$$

Hence, $E''(\theta) = -E(\theta)$ with $E(0) = 1$ and $E'(0) = 0$, so we know $E = \cos$; similarly, $F''(\theta) = -F(\theta)$ with $F(0) = 0$ and $F'(0) = 1$, so $F = \sin$.

Exercise 4.4.15 *Show that $f(t) = \operatorname{cis} \omega t$ is a solution of $f'(t) = i\omega f(t)$ with initial condition $f(0) = 1$.*

The ordinary (real) exponential function, $f(x) = e^x$ is the unique solution to the differential equation $f'(x) = f(x)$ with the initial condition $f(0) = 1$. Further, if $f(t) = ce^{\rho t}$, then $f'(t) = \rho f(t)$ and $f(0) = c$. This is the fundamental equation of exponential growth (or decay), which says that the increase (or decrease) in a quantity is proportional to the current quantity. The real number c is the initial quantity and the real number ρ is the rate of growth (for $\rho > 0$) or decay (for $\rho < 0$).

The foregoing is still true in the system of complex numbers: $f(z) = e^z$ is the unique solution of $f'(z) = f(z)$ with $f(0) = 1$. More generally, for $w \in \mathbb{C}$, ce^{wt} is the unique solution of $f'(t) = wf(t)$ with initial condition $f(0) = c$ (a complex number). The complex number c represents the initial state of the system, comprising a magnitude and phase. However, the meaning of the complex "rate" w requires some explanation.

Write w in rectangular form, $w = \rho + i\omega$. Then the exponential trajectory e^{wt} is seen to be a product of an exponential change in magnitude and a periodic cycle:

$$e^{wt}c = e^{(\rho+i\omega)t}c = e^{\rho t+i\omega t}c = e^{\rho t}e^{i\omega t}c$$

Thus $w = \rho + i\omega$ defines a rate of exponential change ρ and an angular frequency ω.

The parameter w in e^{wt} is sometimes called a *complex frequency*, since both its components are rates and its imaginary component is a rate of rotation. As we will see in Ch. **??**, the "poles and zeros" of filters, which determine their behavior, are complex frequencies. Further, we will see that many systems can be reduced to a sum of complex exponentials, and are thus completely characterized by a set of complex frequencies.

The two components of a complex frequency can be termed its linear rate and its angular frequency. Therefore, we can say that many systems are a superposition of elementary systems, each determined by a linear rate and an angular frequency. In this sense, rectilinear and circular motion are the two primary motions from which almost all complex motions are composed.

Remark 4.4.11 *It is interesting that Aristotle, based on Plato's teachings, distinguished two fundamental motions: rectilinear and circular. This is precisely what we have in a complex frequency: if it is real, we have rectilinear motion; if it is imaginary, we have circular motion. Aristotle said that change in the "sublunary phenomena" (i.e. on the earth) are characterized by rectilinear motion (e.g. a dropped object), whereas the "celestial phenomena" (i.e. in the heavens) are characterized by circular motion (e.g. the motion of the stars). Newton's accomplishment was to show that a single law accounted for both kinds of motion (terrestrial and celestial).*

We may further subdivide the kinds of change based on the signs of the rates: If $w = \rho > 0$ we have an increase; if $w = \rho < 0$ we have a decrease; if $w = i\omega \neq 0$ we have a rotation (counterclockwise for $\omega > 0$, clockwise for $\omega < 0$). If $w = \rho + i\omega$, then we have a combination of rectilinear and circular motion (a spiral outward or inward).

It will be worthwhile to look at these possibilites from the perspective of the differential equation $f'(t) = wf(t)$ or, more compactly, $\dot{z} = wz$. As before, let the initial condition be $z(0) = c$, a complex number.

First suppose $w = \rho$ is real; then the differential equation is $\dot{z} = \rho z$, which means that the change in z is in the same direction as z (for $\rho > 0$), or in the opposite direction (for $\rho < 0$). (Note that $\delta z = \rho z$ is a little vector parallel, or antiparallel, to z; when added to z it increases or decreases its length, but leaves its direction unchanged.) This sort of process causes z to move rectilinearly at an exponential rate: $z(t) = e^{\rho t}c$. Thus the initial state c grows or shrinks exponentially in time. We can see this clearly if we write the initial state in polar form, $c = ae^{i\phi}$; then $z(t) = ae^{\rho t} \times e^{i\phi}$; that is, the angle is independent of time.

Exercise 4.4.16 *Draw z, δz and $z + \delta z$ in this case.*

Next suppose $w = i\omega$ is imaginary; then the differential equation is $\dot{z} = i\omega z$. Recall that multiplication by i is equivalent to a counterclockwise rotation through 90°. Therefore, $\delta z = i\omega z$ can be thought of as a little vector perpendicular to the end of z; it points in a counterclockwise direction for $\omega > 0$ and clockwise for $\omega < 0$. When added to z it causes it to rotate (counterclockwise or clockwise) without changing its length. This sort of process causes z to move circularly, $z(t) = e^{i\omega t}c$. Thus the initial state c rotates periodically with constant magnitude. Putting $c = ae^{i\phi}$ we have, $z(t) = ae^{i(\omega t+\phi)}$; the magnitude is constant a, but the rotation starts with a phase angle ϕ.

Exercise 4.4.17 *Draw z, δz and $z + \delta z$.*

In the general case $w = \rho + i\omega$, we have $\delta z = \rho z + i\omega z$, which is a composite of motion ρz parallel to z and motion $i\omega z$ perpendicular to the end of z. In this case we get a combination of exponential change and rotation, $z(t) = e^{\rho t} e^{i\omega t} c$. If we write the initial state in polar form, $c = ae^{i\phi}$, then

$$z(t) = ae^{\rho t} \times e^{i(\omega t + \phi)}.$$

We see the initial magnitude a changing exponentially by $e^{\rho t}$ and the initial phase angle ϕ rotating by ωt.

Exercise 4.4.18 *Draw z, δz and $z + \delta z$.*

4.4.3 Complex Logarithms

4.4.3.1 Definition

Since the exponential of a complex number scales the real part exponentially to give the magnitude, and converts the imaginary part into a phase angle, we would expect the logarithm of a complex number to reverse this process, deriving the real part from the logarithm of the magnitude and the imaginary part from the phase angle. That is, since $\exp(x + iy) = e^x e^{iy}$, we expect $\ln(e^x e^{iy}) = x + iy$, or equivalently:

$$\ln(re^{i\theta}) = \ln r + i\theta.$$

This is basically correct, but there are some complications we must consider.

The basic problem is that the complex exponential is a periodic function; therefore it is not one-to-one, and so it does not have a unique inverse. In particular, we can see that

$$\ln(re^{i\theta}) = \ln r + i(\theta + m2\pi),$$

for $m = 0, \pm 1, \pm 2, \ldots$. There are several ways we can deal with this.

First we may choose to restrict the angle to lie in a particular range, such as $[0, 2\pi)$ or $[-\pi, \pi)$. Thus we may talk of the *principal* value of the logarithm, as we talk of the principal value of the arcsine, arccosine, etc. (Often the principal value of the logarithm is written "Ln," just as the principal value of the arcsine is written "Arcsin," etc.) This convention has the disadvantage that the identity $\ln(e^{i\theta}) = i\theta$ does not hold unless θ is restricted to the chosen range.

Second, we may simply accept that the logarithm is a *multiple-valued function*; they are not unknown in mathematics and its applications, for example we have $f(x) = \pm\sqrt{x}$ and $f(x) = \sin^{-1} x$. To use multiple-valued functions without encountering contradictions, it's necessary to restrict attention to a particular value, as determined by context, stipulation, or constraints of the application. In this case we can write $\ln(e^{i\theta}) = i\theta$, provided it's understood that the appropriate value of the logarithm must be used.

There is a third, more formal but nevertheless interesting solution, which will be discussed in Section 4.4.3.3.

Exercise 4.4.19 *Show* $e^{\ln z} = z$*, for any of these interpretations of the complex logarithm.*

Exercise 4.4.20 *Show*

$$\ln(zw) = \ln z + \ln w \pmod{2\pi}.$$

4.4.3.2 Geometrical Interpretations

For any integral values of m, observe that

$$e^{x+iy} = e^{x+i(y\pm 2\pi m)}.$$

Hence the values of e^{x+iy} repeat at vertical intervals of 2π. Therefore, if we restrict attention to any infinitely wide "band" of height 2π, the logarithm will be single valued. These bands (which need not have their boundaires at multiples of π or other "reasonable" places) are called *branches* of the complex logarithm. Therefore, if we restrict attention to z in a single branch, we will have $\ln(e^z) = z$.

It will strengthen our intuitive understanding of the complex exponential and logarithm to look at how they transform various subsets of the complex plane.

First, observe that the exponential function maps a branch of the logarithm onto the entire complex plane except for the origin (since $e^z = 0$ has no solution, and so $\ln 0$ is undefined). Conversely, the logarithm maps the complex plane (minus the origin) onto its chosen branch.

4.4.3.3 Riemann Surfaces

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4.4.3.4 Complex Powers

With the complex exponential and logarithm we can define arbitrary powers of complex numbers.

Definition 4.4.1 (Complex Powers) *If z and w are complex numbers ($z \neq 0$), then z^w is defined $z^w = e^{w \ln z}$.*

Remark 4.4.12 *The complex power is multiple-valued because it is defined in terms of the complex logarithm. Therefore it's necessary to restrict attention, by context or stipulation, to a particular branch of the function (either the logarithm or the power).*

Proposition 4.4.6 *The power z^w is single valued if and only if w is an integer.*

Proposition 4.4.7 *If $w = p/q$ is a rational number in lowest terms, then z^w has exactly q values, namely the q principal q-roots of z^p.*

Proposition 4.4.8 *If w is irrational real or complex, then z^w has an infinity of values differing by $e^{2\pi m w i}$.*

Remark 4.4.13 *We have already seen (Prop. 4.4.5) that a complex number has n principal n-th roots. This is consistent with the definition of the n-th root in terms of complex powers, restricted to a branch of the logarithm:*

$$\sqrt[n]{z} = z^{1/n} = e^{(\ln z)/n}.$$

If $z = re^{i\theta}$, then, for $m = 0, 1, \ldots, n-1$,

$$\sqrt[n]{z} = \sqrt[n]{r}\; e^{i(\theta + 2\pi m)/n} = \sqrt[n]{r}\; \operatorname{cis}\left(\frac{\theta}{n} + 2\pi\frac{m}{n}\right).$$

Notice that the roots have equal magnitude and angles evenly distributed around a cycle at angles $2\pi\theta/n$ and beginning at θ/n.

Exercise 4.4.21 *Plot the principal values of $\sqrt[6]{-64i}$.*

4.5 Hyperbolic Geometry

4.5.1 Hyperbolic Functions

4.5.1.1 Hyperbolic Angles

We have seen an intimate connection between the complex exponential and the (circular) trigonometric functions (sine, cosine, etc.); in this section we will explore an equally intimate connection with the *hyperbolic* trigonometric functions. First, we review the derivation of the circular functions.

Draw a circle $x^2 + y^2 = r^2$ and draw a radius at angle θ above the X-axis (Fig. 4.1, p. 42). Drop perpendiculars x and y to the X- and Y-axes. We know from trigonometry that

$$\cos\theta = x/r, \quad \sin\theta = y/r, \quad \tan\theta = y/x.$$

Finally, we have seen (Rem. 4.3.1) that the radian measure of an angle is twice the ratio of the included area to the radius squared.

Now we will undertake a similar construction, but based on the hyperbola rather than the circle; our first task is to define an appropriate measure of angles, in *hyperbolic radians*. Consider the (equilateral) hyperbola $x^2 - y^2 = r^2$, which has its arcs lying within the left and right half planes. Draw a ray from the origin at an angle of less than 45° from the X-axis, so that it intersects the right-hand half of the hyperbola. (We will deal later with angles greater than 45°.) As we did with the circular angle, we measure the hyperbolic angle by the area bounded by the curve between the radius and the X-axis; in particular the hyperbolic radian measure κ will be the ratio of twice the area to the square radius.

Proposition 4.5.1 *Suppose a radius intersects the hyperbola $x^2 - y^2 = r^2$ at the point (x, y). Then the hyperbolic radian measure of the angle of the ray is $\ln\left(\frac{x+y}{r}\right)$.*

Exercise 4.5.1 *To determine this, first show that the area B under the curve of the hyperbola out to x is*

$$B = \frac{xy}{2} - \frac{r^2}{2}\ln\left(\frac{x+y}{r}\right).$$

Exercise 4.5.2 *Show that the required area*

$$A = \frac{r^2}{2} \ln\left(\frac{x+y}{r}\right)$$

by subtracting B from the area of the triangle (x, y, r).

It then follows that the angle in hyperbolic radians is $\kappa = 2A/r^2 = \ln\left(\frac{x+y}{r}\right)$.

Now, just as for the circular functions, we will define the hyperbolic functions in terms the ratios x/r, y/r and y/x. For simplicity, use a unit circle, so $r = 1$ and $\kappa = \ln(x + y)$. Then,

$$\cosh\kappa = x, \quad \sinh\kappa = y, \quad \tanh\kappa = y/x.$$

We therefore have two equations in two unknowns:

$$1 = x^2 - y^2, \tag{4.5}$$
$$\kappa = \ln(x + y). \tag{4.6}$$

Exercise 4.5.3 *Show that the solutions are*

$$x = \frac{e^\kappa + e^{-\kappa}}{2}, \qquad y = \frac{e^\kappa - e^{-\kappa}}{2}$$

(Thus we can "solve triangles" with the hyperbolic functions as well as with the circular, except that we don't have protractors for measuring hyperbolic angles!) We have proved:

Proposition 4.5.2

$$\cosh\kappa = \frac{e^\kappa + e^{-\kappa}}{2}, \tag{4.7}$$
$$\sinh\kappa = \frac{e^\kappa - e^{-\kappa}}{2}, \tag{4.8}$$
$$\tanh\kappa = \frac{e^\kappa - e^{-\kappa}}{e^\kappa + e^{-\kappa}}. \tag{4.9}$$

These formulas are similar to the corresponding Eq. 4.3 for the circular sine and cosine, to which they should be carefully compared. The preceding derivation only applies to angles in the first *octant* ($0° - 45°$). However, by allowing x and y to be negative, it is automatically extended to all angles within 45° of the X-axis. It is extended to angles within 45° of the Y-axis by, in effect, duplicating the above derivation with the hyperbola $y^2 - x^2 = 1$, which has its arc in the upper and lower halfplanes. That is, interchange x and y.

The equations Eq. 4.9 are true in all octants, and in fact are often stipulated as the definition of the functions. In particular, although we have justified these equations on the basis of a real-valued hyperbolic angle, we can use them to define the hyperbolic functions for any complex argument (just as can be done with the exponential formulas for the circular functions, Eq. 4.3).

4.5.1.2 Hyperbolic Functions

Exercise 4.5.4 *Explore and discuss the domain and range of the hyperbolic sine, cosine and tangent over the reals; sketch their shapes (don't plot by computer; use the hyperbolic law of triangles).*

Exercise 4.5.5 *Prove the following symmetry properties: First, the hyperbolic cosine (like the circular cosine) is an* even function, *that is,* $\cosh(-\kappa) = \cosh\kappa$. *Second, the hyperbolic sine (like the circular sine) is an* odd function, *that is,* $\sinh(-\kappa) = -\sinh\kappa$. *As a consequence, the hyperbolic tangent (like the circular tangent) is also odd,* $\tanh(-\kappa) = -\tanh\kappa$.

Exercise 4.5.6 *Prove* $\cosh^2\kappa - \sinh^2\kappa = 1$. *What is the corresponding property of the circular functions?*

Exercise 4.5.7 *Prove*

$$\tanh(\kappa + \lambda) = \frac{\tanh\kappa + \tanh\lambda}{1 + \tanh\kappa\tanh\lambda}.$$

What is the corresponding property of circular functions?

Exercise 4.5.8 *Prove* $\operatorname{sech}^2\kappa = 1 - \tanh^2\kappa$, *where* $\operatorname{sech}\kappa = 1/\cosh\kappa$. *What is the correponding circular property?*

Exercise 4.5.9 *Prove* $\operatorname{csch}^2\kappa = \coth^2\kappa - 1$, *where* $\operatorname{csch}\kappa = 1/\sinh\kappa$ *and* $\coth\kappa = 1/\tanh\kappa$. *What is the corresponding circular property.*

4.5.2 Special Relativity Theory

In this section we will consider briefly special relativity since (1) it illustrates the use of a mixed real/imaginary coordinate system, (2) it makes use of hyperbolic geometry and (3) it suggests ways of treating space and time together, which has relevance to wavelet processing and spatiotemporal information processing in the brain (see my report, "Gabor Representations of Spatiotemporal Visual Images").

4.5.2.1 The Fundamental Invariance

Special relativity, which deals with the geometry of spacetime, is easier to understand by comparison with the geometry of ordinary space. First, notice that in ordinary space certain properties are dependent on the coordinate system we use, whereas others are not. For example, the x and y coordinates of a point (or vector) depend on the choice of axes, since they are projections of that point (or vector) onto the axes.

Exercise 4.5.10 *Diagram this situation.*

On the other hand, the distance between points (or the length of a vector) is independent of the coordinate system. Thus, if (x, y) and (x', y') are the coordinates of the same vector in two different coordinate systems, we can assert the *invariant* $x^2 + x^2 = x'^2 + y'^2$. We say that length is *invariant under a transformation of coordinates.*

In ordinary space there are two different ways we can measure the inclination of a line. If we measure it by slope, then the measure depends on the coordinate system, since the slope is y/x, which quantities are not invariant.

Exercise 4.5.11 *Diagram this situation.*

Further, slopes are not additive: if m and m' are the slopes of the same line in two different coordinate system, and μ is the slope of the primed system with respect to the unprimed, we might expect $m = \mu + m'$, but this is not the case.

Exercise 4.5.12 *In fact, the law of combination is:*

$$m = \frac{\mu+m'}{1-\mu m'} \text{ or } m' = \frac{m-\mu}{1+m\mu}.$$

Does this look familiar? Derive it by trigonometry. Notice also that $m \approx \mu + m'$ *if* $\mu \approx 0$*; that is, if the coordinate systems deviate only slightly from each other, then slopes are approximately additive.*

On the other hand, ordinary (circular) angles *are* additive, since they are invariant under coordinate transformation. Therefore, if a line has angles a and a' with respect to the X-axes of two coordinate systems, and α is the angle of the primed X′-axis to the X-axis, then $a' = \alpha + a$.

Exercise 4.5.13 *Diagram this situation.*

In relativity, events are located in four-dimensional spacetime; they have coordinates (x, y, z, t). Now we will make several convenient assumptions. First, since relativistic effects occur in the direction of motion, and not perpendicular to the direction of motion, we will restrict our attention to a single space axis s, oriented in the direction of motion; thus spacetime coordinates will take the form (s, t). This will make spacetime geometry easier to visualize and draw, and will simplify the mathematical notation.

Second, since time is an axis like the other three, we will measure them all in the same units, meters, which will simplify the formulas. (Imagine the needless complexity that would result from measuring north- south distances in miles and east-west distance in kilometers.) This raises the question of how to convert seconds to meters; what is the conversion factor? It turns out that it is the speed of light, $c \approx 3 \times 10^8$ m/s. We will see that this is not an arbitrary choice, but is in fact fundamental to the fabric of spacetime. If T is time in seconds and t is time in meters, then $t = cT$.

Finally, in accord with the measurement of time in meters, velocity becomes a pure number (meters/meter), for which relativity theory uses the symbol β. If $V = s/T$ is time in ordinary units, we can see that $\beta = s/t = s/(cT) = V/c$. Thus β can also be interpreted as velocity relative to the speed of light. By "natural units" I will mean the measurement of time in meters and velocity as a pure number.

Remark 4.5.1 *In our lives we range cover a vast distance along the time axis compared to our range on the spatial axes. Since there are about* $\pi \times 10^7$ *seconds in a year, we go about* $\pi c \times 10^7 \approx 3\pi \times 10^{15} \approx 10^{16}$ *meters in a year (i.e. one light-year). In our lifetimes we cover about* 7×10^{17} *meters on the time axis (that is, about 70 light-years, something between the distances to Aldebaràn and to Regulus). In the same amount of time the solar system* *moves about* 7×10^{14} *m. relative to the cosmic background radiation (since our motion with respect to it is about* 3×10^5 *m/s in the direction of Virgo). It is this large discrepency, a ratio of* 10^3*, between our mobility in time and space that leads to the undetectability of relativistic effects under ordinary conditions. Therefore, our average velocity, in natural units, is* $\beta \approx 10^{-3}$ *(i.e. speed relative to background radiation divided by speed of light).*

Although we measure it in spatial units, the time axis is not just another space axis; indeed we may say that time is imaginary with respect to the spatial axes, since one consequence of the relativity postulates is that the fundamental invariant is $s^2 + (it)^2 = s^2 - t^2$.

Remark 4.5.2 *This invariant follows from the first postulate of relativity theory, which says that the velocity of light in a vacuum is the same in all reference frames. To see this suppose that a reflective object is moving to the right past us at a velocity* β*. When it is directly opposite us at a distance of* r*, suppose that it is struck by light from source distance* s *to our left. The distance, as measured in our reference frame, traveled by the light is* $d = \sqrt{r^2 + s^2}$*, so is the time, in our frame, that it took to travel it (since the speed of light = 1 in natural units):* $t = \sqrt{r^2 + s^2}$*. Within the reference frame of the reflective object, however, the source appears to be a distance* s' *to the left, so the distance the light travelled is* $d' = \sqrt{r^2 + s'^2}$*; likewise the time is* $t' = \sqrt{r^2 + s'^2}$*. Now observe:*

$$\begin{aligned} t^2 - s^2 &= (r^2 + s^2) - s^2 = r^2, \\ t'^2 - s'^2 &= (r^2 + s'^2) - s'^2 = r^2. \end{aligned}$$

We see that $t^2 - s^2 = t'^2 - s'^2$*.*

This quantity, $s^2 - t^2$, which is invariant under a change between reference frames in relative motion, is called the *spacetime interval* between two events; it is analogous to the Euclidean distance $x^2 + y^2$, which is invariant under change of the spatial coordinate system.

Spacetime intervals can be classified according to whether $s^2 - t^2$ is positive, negative or zero. If it is positive, the interval is called *space-like* and the *proper distance* σ is defined $\sigma^2 = s^2 - t^2$. If it is negative, the interval is called *time-like* and the *proper time* τ is defined $\tau^2 = t^2 - s^2$. If the interval is zero, it is called *light-like*. We will see that time-like intervals can be crossed by *subluminary signals* (signals travelling less than the speed of light), and light-like intervals can be crossed only by things travelling at the speed of light. Space-like intervals could be crossed only by things travelling faster than light, which, so far as physics has been able to establish, do not exist. Therefore, space-like intervals are causally independent; causality can operate only across time-like and light-like intervals (i.e., those for which $t^2 - s^2 \geq 0$). Therefore, we will restrict our attention to this case (without loss of generality, however).

Exercise 4.5.14 *The invariant* $t^2 - s^2 = \text{constant}$ *should remind you of an identity that you have seen recently. What does it suggest about the formal relation of the quantities* t *and* s*?*

4.5.2.2 Meaning of the Hyperbolic Angle

The invariance of spacetime interval means that, for a given pair of events, $\tau^2 = t^2 - s^2$ is constant, no matter what their distance separation s and time separation t in a given reference frame. That is, the possible s and t measurements in various reference frames is constrained by $\tau^2 = t^2 - s^2$. This means that the possible (s, t) pairs lie on an equilateral hyperbola, whose arcs lie in the positive and negative t halfplanes. From this we see, by the law of triangles for hyperbolas, that (for some hyperbolic angle κ):

$$\begin{aligned} s &= \tau \sinh \kappa, \\ t &= \tau \cosh \kappa. \end{aligned}$$

From these, the invariance of the spacetime interval follows from the properties of the hyperbolic functions:

$$t^2 - s^2 = \tau^2(\cosh^2 \kappa - \sinh^2 \kappa) = \tau^2.$$

Now we must consider the meaning of the hyperbolic angle κ. Observe that the velocity can be written in terms of hyperbolic functions:

$$\beta = \frac{s}{t} = \frac{\tau \sinh \kappa}{\tau \cosh \kappa} = \tanh \kappa.$$

Thus, $\kappa = \operatorname{arctanh} \beta$, and so it is called the *velocity parameter*; we may say that the velocity parameter is the hyperbolic arctangent of the velocity (in natural units). The significance of the velocity parameter is that it measures the hyperbolic angle between the time axes in the two reference frames. Just as a slope m is related to a corresponding angle θ by the circular tangent, $m = \tan\theta$, so a velocity is related to a corresponding velocity parameter by the hyperbolic tangent, $\beta = \tanh\kappa$. The appearance of the circular functions in spatial rotations is a consequence of the isotropy of x and y; the appearance of the hyperbolic functions in spacetime transformations is a consequence of the anisotropy of s and it (i.e., time is imaginary with respect to space).

4.5.2.3 Comparison of Lorentz & Galilean Transforms

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4.6 References

Sources for this chapter include Boas (1983, ch. 2), Bobrow (1981, §8.3), Marsden (1973, ch. 1), and Taylor & Wheeler (1966, pp. 22–6, 39–59, 92–4).