B.3 Uncertainty principle

B.3.a Informally

¶1. **Heisenberg Uncertainty Principle:** The uncertainty principle states a lower bound on the precision with which certain pairs of variables can be measured.

¶2. **Conjugate variables:** These are such pairs as position and momentum, and energy and time.

For example, the same state can be represented by the wave function $\psi(x)$ as a function of space and by $\phi(p)$ as a function of momentum.

¶3. Example: $\Delta x \, \Delta p \geq \hbar/2$.

$\hbar = h/2\pi$, where h is Planck's constant. They are defined $E = h\nu$ (Hertz, or cycles per second) and $E = \hbar\omega$ (radians per second).

¶4. **Observer effect:** "While it is true that measurements in quantum mechanics cause disturbance to the system being measured, this is most emphatically *not* the content of the uncertainty principle."[1]

(The disturbance is called the *observer effect*.)

¶5. Typically the uncertainty principle is a result of the variables representing measurements in two bases that are Fourier transforms of each other.
For example, time and energy are conjugate; note $\psi(t)$ and $\phi(E) = \Psi(\nu)$, where $E = h\nu$. (For momentum, the de Broglie relation is $p\lambda = h$, where λ = wavelength, or $p = \hbar k$, where $k = 2\pi/\lambda$ is the angular wavenumber, the number of wavelengths per 2π units of distance.)

¶6. **Example:** Consider an audio signal $\psi(t)$ and its Fourier transform $\Psi(\nu)$. Note that ψ is a function of time, with dimension t, and its spectrum Ψ is a function of frequency, with dimension t^{-1}.
They are recirocals of each other, and that is always the case with Fourier transforms.

[1]NC 89.

¶7. For more details on this, including an intuitive explanation, see FFC, ch. 6.)

¶8. **Non-commutative operators:** More generally, the observables are represented by Hermitian operators P, Q that do not commute. That is, to the extent they do not compute, to that extent you cannot measure them both (because you would have to do either PQ or QP, but they do not give the same result).

¶9. Best interpretation: If you set up the experiment multiple times, and measure the outcomes, you will find

$$2\,\Delta P\,\Delta Q \geq |\langle [P,Q] \rangle|,$$

where P and Q are conjugate observables.

¶10. Note that this is a *purely mathematical* result. Any system obeying the QM postulates will have uncertainty principles for every pair of non-commuting observables.

B.3.b Formally

Optional! The following is from FFC, ch. 5.

¶1. **Definition B.1 (commutator)** *If $L, M : \mathcal{H} \to \mathcal{H}$ are linear operators, then their* commutator *is defined:*

$$[L, M] = LM - ML. \tag{III.2}$$

Remark B.1 *In effect, $[L, M]$ distills out the non-commutative part of the product of L and M. If the operators commute, then $[L, M] = \mathbf{0}$, the identically zero operator. Constant-valued operators always commute ($cL = Lc$), and so $[c, L] = \mathbf{0}$.*

¶2. **Definition B.2 (anti-commutator)** *If $L, M : \mathcal{H} \to \mathcal{H}$ are linear operators, then their* anti-commutator *is defined:*

$$\{L, M\} = LM + ML. \tag{III.3}$$

If $\{L, M\} = \mathbf{0}$, we say that L and M anti-commute*, $LM = -ML$.*

¶3. See B.1.c (p. 62) for the justification of the following definitions.

Definition B.3 (mean of measurement) *If M is a Hermitian operator representing an observable, then the mean value of the measurement of a state $|\psi\rangle$ is*

$$\langle M \rangle = \langle \psi \mid M \mid \psi \rangle.$$

¶4. **Definition B.4 (variance and standard deviation of measurement)** *If M is a Hermitian operator representing an observable, then the variance in the measurement of a state $|\psi\rangle$ is*

$$\mathrm{Var}\{M\} = \langle (M - \langle M \rangle^2) \rangle = \langle M^2 \rangle - \langle M \rangle^2.$$

As usual, the standard deviation ΔM *of the measurement is defined*

$$\Delta M = \sqrt{\mathrm{Var}\{M\}}.$$

¶5. **Proposition B.1** *If L and M are Hermitian operators on $\mathcal{H}$ and $|\psi\rangle \in \mathcal{H}$, then*

$$4\langle \psi \mid L^2 \mid \psi \rangle \, \langle \psi \mid M^2 \mid \psi \rangle \geq |\langle \psi \mid [L, M] \mid \psi \rangle|^2 + |\langle \psi \mid \{L, M\} \mid \psi \rangle|^2.$$

More briefly, in terms of average measurements,

$$4\langle L^2 \rangle \langle M^2 \rangle \geq |\langle [L, M] \rangle|^2 + |\langle \{L, M\} \rangle|^2.$$

Proof: Let $x + iy = \langle \psi \mid LM \mid \psi \rangle$. Then,

$$\begin{aligned}
2x &= \langle \psi \mid LM \mid \psi \rangle + (\langle \psi \mid LM \mid \psi \rangle)^* \\
&= \langle \psi \mid LM \mid \psi \rangle + \langle \psi \mid M^\dagger L^\dagger \mid \psi \rangle \\
&= \langle \psi \mid LM \mid \psi \rangle + \langle \psi \mid ML \mid \psi \rangle \quad \text{since } L, M \text{ are Hermitian} \\
&= \langle \psi \mid \{L, M\} \mid \psi \rangle.
\end{aligned}$$

Likewise,

$$\begin{aligned}
2iy &= \langle \psi \mid LM \mid \psi \rangle - (\langle \psi \mid LM \mid \psi \rangle)^* \\
&= \langle \psi \mid LM \mid \psi \rangle - \langle \psi \mid ML \mid \psi \rangle \\
&= \langle \psi \mid [L, M] \mid \psi \rangle.
\end{aligned}$$

Hence,

$$\begin{aligned} |\langle \psi \mid LM \mid \psi \rangle|^2 &= 4(x^2 + y^2) \\ &= |\langle \psi \mid [L, M] \mid \psi \rangle|^2 + |\langle \psi \mid \{L, M\} \mid \psi \rangle|^2. \end{aligned}$$

Let $|\lambda\rangle = L|\psi\rangle$ and $|\mu\rangle = M|\psi\rangle$. By the Cauchy-Schwarz inequality, $\|\lambda\| \, \|\mu\| \geq |\langle \lambda \mid \mu \rangle|$ and so $\langle \lambda \mid \lambda \rangle \, \langle \mu \mid \mu \rangle \geq |\langle \lambda \mid \mu \rangle|^2$. Hence,

$$\langle \psi \mid L^2 \mid \psi \rangle \, \langle \psi \mid M^2 \mid \psi \rangle \geq |\langle \psi \mid LM \mid \psi \rangle|^2.$$

The result follows.

□

¶6. **Proposition B.2** *Prop. B.1 can be weakened into a more useful form:*

$$4\langle \psi \mid L^2 \mid \psi \rangle \, \langle \psi \mid M^2 \mid \psi \rangle \geq |\langle \psi \mid [L, M] \mid \psi \rangle|^2,$$

or $4\langle L^2 \rangle \langle M^2 \rangle \geq |\langle [L, M] \rangle|^2$

¶7. **Proposition B.3 (uncertainty principle)** *If Hermitian operators P and Q are measurements (observables), then*

$$\Delta P \, \Delta Q \geq \frac{1}{2} |\langle \psi \mid [P, Q] \mid \psi \rangle|.$$

That is, $\Delta P \;\; \Delta Q \geq |\langle [P, Q] \rangle|/2$. So the product of the variances is bounded below by the degree to which the operators do not commute.

Proof: Let $L = P - \langle P \rangle$ and $M = Q - \langle Q \rangle$. By Prop. B.2 we have

$$\begin{aligned} 4 \, \mathrm{Var}\{P\} \, \mathrm{Var}\{Q\} &= 4\langle L^2 \rangle \langle M^2 \rangle \\ &\geq |\langle [L, M] \rangle|^2 \\ &= | \, \langle [P - \langle P \rangle, Q - \langle Q \rangle] \rangle \, |^2 \\ &= |\langle [P, Q] \rangle|^2. \end{aligned}$$

Hence,

$$2 \, \Delta P \Delta Q \geq |\langle [P, Q] \rangle|$$

□