B.5 Entanglement

¶1. Suppose that $\mathcal{H}'$ and $\mathcal{H}''$ are the state spaces of two systems. Then $\mathcal{H} = \mathcal{H}' \otimes \mathcal{H}''$ is the state space of the *composite system*.

¶2. For simplicity, suppose that both spaces have the basis $\{|0\rangle, |1\rangle\}$. Then $\mathcal{H}' \otimes \mathcal{H}''$ has basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.
Recall that $|01\rangle = |0\rangle \otimes |1\rangle$, etc.

¶3. Arbitrary elements of $\mathcal{H}' \otimes \mathcal{H}''$ can be written in the form

$$\sum_{j,k=0,1} c_{jk}|jk\rangle = \sum_{j,k=0,1} c_{jk}\ |j'\rangle \otimes |k''\rangle.$$

¶4. Sometimes the state of the composite systems can be written as the tensor product of the states of the subsystems, $|\psi\rangle = |\psi'\rangle \otimes |\psi''\rangle$. Such a state is called a *separable*, *decomposable* or *product state*.

¶5. In other cases the state cannot be decomposed, in which case it is called an *entangled state*

¶6. **Bell entangled state:** For an example of an entangled state, consider the *Bell state* Φ^+, which might arise from a process that produced two particles with opposite spin (but without determining which is which):

$$\beta_{01} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \stackrel{\text{def}}{=} \Phi^+. \tag{III.4}$$

(The notations β_{01} and Φ^+ are both used.)
Note that the states $|01\rangle$ and $|10\rangle$ both have probability 1/2.

¶7. Such a state might arise from a process that emits two particles with opposite spin angular momentum in order to preserve conservation of spin angular momentum.

¶8. To show that it's entangled, we need to show that it cannot be decomposed

$$\beta_{01} \stackrel{?}{=} (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle).$$

Multiplying out the RHS yields:

$$a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle.$$

Therefore we must have $a_0b_0 = 0$ and $a_1b_1 = 0$. But this implies that either $a_0b_1 = 0$ or $a_1b_0 = 0$ (as opposed to $1/\sqrt{2}$), so the decomposition is impossible.

¶9. **Decomposable state:** Consider: $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Writing out the product $(a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$ as before, we require $a_0b_0 = a_0b_1 = a_1b_0 = a_1b_1 = \frac{1}{2}$. This is satisfied by $a_0 = a_1 = b_0 = b_1 = \frac{1}{\sqrt{2}}$.

¶10. **Bell states:** In addition to Eq. III.4, the other three Bell states are defined:

$$\beta_{00} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \stackrel{\text{def}}{=} \Psi^+, \tag{III.5}$$

$$\beta_{10} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \stackrel{\text{def}}{=} \Psi^-, \tag{III.6}$$

$$\beta_{11} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \stackrel{\text{def}}{=} \Phi^-. \tag{III.7}$$

¶11. The Ψ states have two identical qubits, the Φ states have opposite. The + superscript indicates they are added, the − that they are subtracted.

¶12. The general definition is:

$$\beta_{xy} = \frac{1}{\sqrt{2}}(|0, y\rangle + (-1)^x|1, \neg y\rangle).$$