B.6   Dynamics

B.6.a   CONTINUOUS TIME

¶1. **Schrödinger equation:** The time evolution of a closed QM system is given by the Schrödinger equation:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle,$$

or more compactly, $i\hbar|\dot\psi\rangle = H|\psi\rangle$.
H is the Hamiltonian of the system (a fixed Hermitian operator).
$\hbar$ is Planck's constant (often absorbed into H).

¶2. Since H is Hermitian, it has a spectral decomposition, $H = \sum_E E|E\rangle\langle E|$, where the normalized $|E\rangle$ are *energy eigenstates* (or *stationary states*) with corresponding energies E.
The lowest energy is the *ground state energy*.

B.6.b   DISCRETE TIME

¶1. Stone's theorem shows that the solution to the Schrödinger equation is:
$$|\psi(t+s)\rangle = e^{-iHt/\hbar}|\psi(s)\rangle.$$

¶2. Define $U(t) \overset{\text{def}}{=} \exp(-iHt/\hbar)$; then $|\psi(t+s)\rangle = U(t)|\psi(s)\rangle$.

¶3. U is unitary (Exer. III.3).

¶4. Hence the evolution of a closed QM system from a state $|\psi\rangle$ at time t to a state $|\psi'\rangle$ at time t' can be described by a unitary operator, $|\psi'\rangle = U|\psi\rangle$.

¶5. For any unitary operator U there is a Hermitian K such that $U = \exp(iK)$ (Exer. III.4).

B.7   No-cloning theorem

¶1. The *No-cloning Theorem* states that it is impossible to copy the state of a qubit.

¶2. On the contrary, assume that we have a unitary transformation U that does the copying, so that $U(|\psi\rangle \otimes |c\rangle) = |\psi\rangle \otimes |\psi\rangle$, where $|c\rangle$ is an arbitrary constant qubit.
That is $U|\psi c\rangle = |\psi\psi\rangle$.

¶3. Suppose $|\psi\rangle = a|0\rangle + b|1\rangle$.

¶4. By the linearity of U:

$$
\begin{aligned}
U\,|\psi\rangle|c\rangle &= U\,(a|0\rangle + b|1\rangle)|c\rangle \\
&= U(a|0\rangle|c\rangle + b|1\rangle|c\rangle) \quad \text{distrib. of tensor prod.} \\
&= U(a|0c\rangle + b|1c\rangle) \\
&= a(U|0c\rangle) + b(U|1c\rangle) \quad \text{linearity} \\
&= a|00\rangle + b|11\rangle \quad \text{copying property.}
\end{aligned}
$$

¶5. By expanding $|\psi\psi\rangle$ we have:

$$
\begin{aligned}
U\,|\psi c\rangle &= |\psi\psi\rangle \\
&= (a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle) \\
&= a^2|00\rangle + ba|10\rangle + ab|01\rangle + b^2|11\rangle.
\end{aligned}
$$

¶6. Note that these two expansions cannot be made equal in general, so no such unitary transformation exists.

¶7. Cloning is possible in the special cases $a = 0, b = 1$ or $a = 1, b = 0$, that is, where we know that we a cloning a pure basis state.

B.8   EPR paradox

¶1. Proposed by Einstein, Podolsky, and Rosen in 1935 to show problems in QM.

¶2. Suppose a source produces an entangled *EPR pair* (or *Bell state*) $\Psi^+ = \beta_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and the particles are sent to Alice and Bob.

¶3. If Alice measures her particle and gets $|0\rangle$, then that collapses the state to $|00\rangle$, and so Bob will have to get $|0\rangle$ if he measures. And likewise if Alice happens to get $|1\rangle$.

¶4. This happens instantaneously (but it does not permit faster-than-light communication).

¶5. **Hidden-variable theories:** One explanation is that there is some internal state in the particles that will determine the result of the measurement. Both particles have the same internal state.
This cannot explain the results of measurements in different bases.
In 1964 John Bell showed that any local hidden variable theory would lead to measurements satisfying a certain inequality (Bell's inequality).
Actual experiments violate Bell's inequality.
Thus local hidden variable theories cannot be correct.

¶6. **Causal theories:** Another explanation is that Alice's measurement affects Bob's (or vice versa, if Bob measures first).
According to relativity theory, in some frames of reference Alice's measurement comes first, and in others, Bob's.
Therefore there is no consistent cause-effect relation.
This is why Alice and Bob cannot use entangled pairs to communicate.