D.2 Simon

Simon's algorithm was presented in Simon, D. (1997), "On the power of quantum computation," *SIAM Journ. Computing*, 26 (5), pp. 1474–83. This presentation follows Mermin's *Quantum Computer Science*, §2.5 (pp. 55–8) [MQCS].

¶1. For breaking RSA we will see that its useful to know the *period* of a function: that p such that $f(x+p) = f(x)$. Simon's problem is a warmup for this.

¶2. **Simon's Problem:** Suppose we are given an unknown function $f : \mathbf{2}^n \rightarrow \mathbf{2}^n$ and we are told that it is *two-to-one*.
This means $f(\mathbf{x}) = f(\mathbf{y})$ iff $\mathbf{x} \oplus \mathbf{y} = \mathbf{p}$ for some fixed $\mathbf{p} \in \mathbf{2}^n$.
The vector $\mathbf{p}$ can be considered the *period* of f, since $f(\mathbf{x} \oplus \mathbf{p}) = f(\mathbf{x})$.

¶3. The problem is to determine the period $\mathbf{p}$ of an unknown f.

¶4. **Classical solution:** Since we don't know anything about f, the best we can do is evaluate it on random inputs.
If we are ever lucky enough to find $\mathbf{x}$ and $\mathbf{x}'$ such that $f(\mathbf{x}) = f(\mathbf{x}')$, then we have our answer, $\mathbf{p} = \mathbf{x} \oplus \mathbf{x}'$.

¶5. On the average you need to do $2^{n/2}$ function evaluations, which is exponential in the size of the input.
For $n = 100$, it would require about $2^{50} \approx 10^{15}$ evaluations. "At 10 million calls per second it would take about three years." [MQCS 55]

¶6. **Quantum algorithm:** We will see that a quantum computer can determine $\mathbf{p}$ with high probability ($> 1 - 10^{-6}$) in about 120 evaluations. At 10 million calls per second, this would take about 12 microseconds!

¶7. **Input superposition:** As before, start by using the Walsh-Hadamard transform to create a superposition of all possible inputs:

$$|\psi_1\rangle \stackrel{\text{def}}{=} H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{\mathbf{x} \in \mathbf{2}^n} |\mathbf{x}\rangle.$$

¶8. **Function evaluation:** Suppose that U_f is the quantum gate array implementing f and recall $U_f|\mathbf{x}\rangle|\mathbf{y}\rangle = |\mathbf{x}\rangle|\mathbf{y} \oplus f(\mathbf{x})\rangle$. Therefore:

$$|\psi_2\rangle \stackrel{\text{def}}{=} U_f|\psi_1\rangle|0\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{\mathbf{x} \in \mathbf{2}^n} |\mathbf{x}\rangle|f(\mathbf{x})\rangle.$$

Therefore we have an equal superposition of corresponding input-output values.

¶9. **Output measurement:** Measure the output register to obtain some $|\mathbf{z}\rangle$.
Since the function is two-to-one, the projection will have a superposition of two inputs:

$$\frac{1}{\sqrt{2}}(|\mathbf{x}_0\rangle + |\mathbf{x}_0 + \mathbf{p}\rangle)|\mathbf{z}\rangle,$$

where $f(\mathbf{x}_0) = \mathbf{z} = f(\mathbf{x}_0 + \mathbf{p})$.

¶10. The information we need is contained in the input register,

$$|\psi_3\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|\mathbf{x}_0\rangle + |\mathbf{x}_0 + \mathbf{p}\rangle),$$

but it cannot be extracted directly.
If we measure it, we will get either $\mathbf{x}_0$ or $\mathbf{x}_0 + \mathbf{p}$, but not both, and we need both to get $\mathbf{p}$.
(We cannot make two copies, due to the no-cloning theorem.)

¶11. Suppose we apply the Walsh-Hadamard transform to this superposition:

$$\begin{aligned} H^{\otimes n}|\psi_3\rangle &= H^{\otimes n}\frac{1}{\sqrt{2}}(|\mathbf{x}_0\rangle + |\mathbf{x}_0 + \mathbf{p}\rangle) \\ &= \frac{1}{\sqrt{2}}(H^{\otimes n}|\mathbf{x}_0\rangle + H^{\otimes n}|\mathbf{x}_0 + \mathbf{p}\rangle). \end{aligned}$$

¶12. Now, recall (¶13, p. 112) that $H^{\otimes n}|\mathbf{x}\rangle = \frac{1}{2^{n/2}}\sum_{\mathbf{y}\in\mathbf{2}^n}(-1)^{\mathbf{x}\cdot\mathbf{y}}|\mathbf{y}\rangle$. Therefore,

$$\begin{aligned} H^{\otimes n}|\psi_3\rangle &= \frac{1}{\sqrt{2}}\left[\frac{1}{2^{n/2}}\sum_{\mathbf{y}\in\mathbf{2}^n}(-1)^{\mathbf{x}_0\cdot\mathbf{y}}|\mathbf{y}\rangle + \frac{1}{2^{n/2}}\sum_{\mathbf{y}\in\mathbf{2}^n}(-1)^{(\mathbf{x}_0+\mathbf{p})\cdot\mathbf{y}}|\mathbf{y}\rangle\right] \\ &= \frac{1}{2^{(n+1)/2}}\sum_{\mathbf{y}\in\mathbf{2}^n}\left[(-1)^{\mathbf{x}_0\cdot\mathbf{y}} + (-1)^{(\mathbf{x}_0+\mathbf{p})\cdot\mathbf{y}}\right]|\mathbf{y}\rangle. \end{aligned}$$

¶13. Note that $(-1)^{(\mathbf{x}_0+\mathbf{p})\cdot\mathbf{y}} = (-1)^{\mathbf{x}_0\cdot\mathbf{y}}(-1)^{\mathbf{p}\cdot\mathbf{y}}$.
Therefore, if $\mathbf{p}\cdot\mathbf{y} = 1$, then the bracketed expression is 0 (since the

terms have opposite sign and cancel).
However, if $\mathbf{p} \cdot \mathbf{y} = 0$, then the bracketed expression is $2(-1)^{\mathbf{x}_0 \cdot \mathbf{y}}$ (since they don't cancel).

¶14. Hence the result of the Walsh-Hadamard transform is

$$|\psi_4\rangle = H^{\otimes n}|\psi_3\rangle = \frac{1}{2^{(n-1)/2}} \sum_{\mathbf{p}\cdot\mathbf{y}=0} (-1)^{\mathbf{x}_0 \cdot \mathbf{y}} |\mathbf{y}\rangle.$$

¶15. **Measurement:** Measuring the input register will collapse it with equal probability into a state $|\mathbf{y}^{(1)}\rangle$ such that $\mathbf{p} \cdot \mathbf{y}^{(1)} = 0$.

¶16. **First equation:** Since we know $\mathbf{y}^{(1)}$, this gives us some information about $\mathbf{p}$, expressed in the equation:

$$y_1^{(1)} p_1 + y_2^{(1)} p_2 + \cdots + y_n^{(1)} p_n = 0 \pmod{2}.$$

¶17. **Iteration:** The quantum computation can be repeated, producing a series of bit strings $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots$ such that $\mathbf{y}^{(k)} \cdot \mathbf{p} = 0$.
From them we can build up a system of n linearly-independent equations and solve for $\mathbf{p}$.
(If you get a linearly-dependent equation, you have to try again.)

¶18. Note that each quantum step (involving one evaluation of f) produces an equation (except in the unlikely case $\mathbf{y}^{(k)} = 0$ or that it's linearly dep.), and therefore determines one of the bits in terms of the other bits.
That is, each iteration reduced the candidates for $\mathbf{p}$ by approximately one-half.

¶19. **Probability:** A mathematical analysis [MQCS App. G] shows that with $n + m$ iterations the probability of having enough information to determine $\mathbf{p}$ is $> 1 - \frac{1}{2^{m+1}}$.
"Thus the odds are more than a million to one that with $n+20$ invocations of $\mathbf{U}_f$ we will learn [$\mathbf{p}$], no matter how large n may be." [MQCS 57]

¶20. **Exponential speedup:** Therefore Simon's problem can be solved in *linear* time on a quantum computer, but requires *exponential* time on a classical computer.