B.5 Superposition

B.5.a Bases

¶1. In QM certain physical quantities are quantized, such as the energy of an electron in an atom.
Therefore an atom might be in certain distinct energy states $|\text{ground}\rangle$, $|\text{first excited}\rangle$, $|\text{second excited}\rangle$, ...

¶2. Other particles might have distinct states such as spin-up $|\uparrow\rangle$ and spin-down $|\downarrow\rangle$.

¶3. In each case these alternative states are orthonormal: $\langle\uparrow|\downarrow\rangle = 0$; $\langle\text{ground} \mid \text{first excited}\rangle = 0$, $\langle\text{ground} \mid \text{second excited}\rangle = 0$, $\langle\text{first excited} \mid \text{second excited}\rangle = 0$.

¶4. In general we may express the same state with respect to different bases, such as vertical or horizontal polarization $|\rightarrow\rangle$, $|\uparrow\rangle$; or orthogonal diagonal polarizations $|\nearrow\rangle$, $|\searrow\rangle$.

B.5.b Superpositions of Basis States

¶1. One of the unique characteristics of QM is that a physical system can be in a superposition of basis states, for example,

$$|\psi\rangle = c_0|\text{ground}\rangle + c_1|\text{first excited}\rangle + c_2|\text{second excited}\rangle,$$

where the c_j are complex numbers, called *(probability) amplitudes*.

¶2. Since $\| |\psi\rangle \| = 1$, we know $|c_0|^2 + |c_1|^2 + |c_2|^2 = 1$.

¶3. With respect to a given basis, a state $|\psi\rangle$ is interchangeable with its vector of coefficients, $\mathbf{c} = (c_0, c_1, \ldots, c_n)^{\mathrm{T}}$. When the basis is understood, we can use $|\psi\rangle$ as a name for this vector.

¶4. **Quantum parallelism:** The ability of a quantum system to be in many states simultaneously is the foundation of *quantum parallelism*.

¶5. **Measurement:** As we will see, when we measure the quantum state

$$c_0|E_0\rangle + c_1|E_1\rangle + \ldots + c_n|E_n\rangle$$

with respect to the $|E_0\rangle, \ldots, |E_n\rangle$ basis, we will get the result $|E_j\rangle$ with probability $|c_j|^2$ and the state will "collapse" into state $|E_j\rangle$.

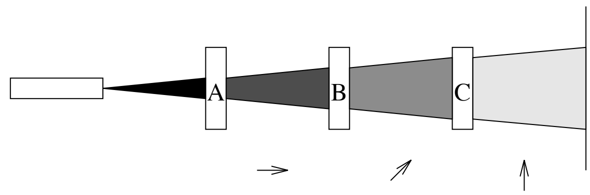

Figure III.4: Fig. from IQC.

¶6. **Qubit:** For the purposes of quantum computation, we usually pick two basis states and use them to represent the bits 1 and 0, for example, $|1\rangle = |\text{ground}\rangle$ and $|0\rangle = |\text{excited}\rangle$.
I've picked the opposite of the "obvious" assignment ($|0\rangle = |\text{ground}\rangle$) just to show that the assignment is arbitrary (just as for classical bits).

¶7. Note that $|0\rangle \neq \mathbf{0}$, the zero element of the vector space, since $\| |0\rangle \| = 1$ but $\|\mathbf{0}\| = 0$. (Thus $\mathbf{0}$ does not represent a physical state.)

B.5.c Photon polarization experiment

See Fig. III.4.

¶1. **Experiment:** Suppose we have three polarizing filters, A, B, and C, polarized horizontally, 45°, and vertically, respectively.

¶2. Place filter A between strong light source and screen. Intensity is reduced by half and light is horizontally polarized.
(Note: intensity would be much less if it allowed only horizontally polarized light through, as in sieve model.)

¶3. Insert filter C and intensity drops to zero. No surprise, since cross-polarized.

¶4. Insert filter B between A and C, and some light (about 1/8 intensity) will return!
Can't be explained by sieve model.

¶5. **Explanation:** A photon's polarization state can be represented by a unit vector pointing in appropriate direction.

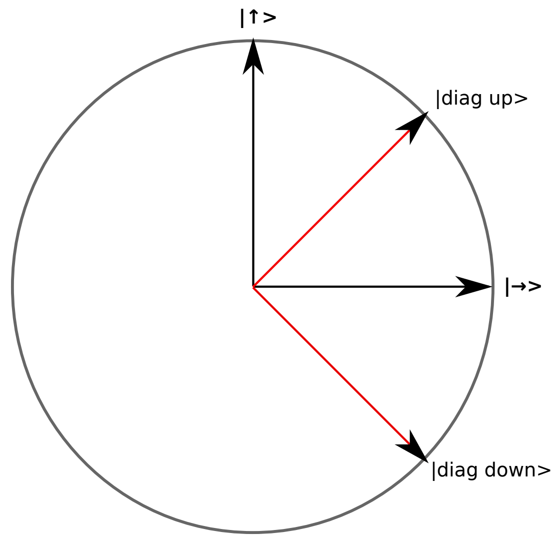

Figure III.5: Alternative polarization bases for measuring photons (black = rectilinear basis, red = diagonal basis). Note $|\nearrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\rightarrow\rangle)$ and $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\nearrow\rangle + |\searrow\rangle)$.

¶6. Arbitrary polarization can be expressed by $a|0\rangle + b|1\rangle$ for any two basis vectors $|0\rangle$, $|1\rangle$, where $|a|^2 + |b|^2 = 1$.

¶7. A polarizing filter measures a state with respect to a basis that includes a vector parallel to polarization and one orthogonal to it.

¶8. The filter A is the projector $|\rightarrow\rangle\langle\rightarrow|$.
To get the probability amplitude, apply to $|\psi\rangle \stackrel{\text{def}}{=} a|\rightarrow\rangle + b|\uparrow\rangle$:

$$\langle\rightarrow|\,\psi\rangle = \langle\rightarrow|(a|\rightarrow\rangle + b|\uparrow\rangle) = a\langle\rightarrow|\rightarrow\rangle + b\langle\rightarrow|\uparrow\rangle = a.$$

So with probability $|a|^2$ we get $|\rightarrow\rangle$. Recall (Eqn. III.1, p. 93):

$$p(|\rightarrow\rangle) = \|\langle\rightarrow|\,\psi\rangle\|^2 = |a|^2.$$

¶9. So if the polarizations are randomly distributed from the source, half will get through with resulting photons all $|\rightarrow\rangle$.
Why 1/2? Note $a = \cos\theta$ and $\langle a^2\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta = \frac{1}{2}$.

¶10. When we insert filter C we are measuring with $\langle\uparrow|$ and the result is 0, as expected.

¶11. **Diagonal filter:** Filter B measures with respect to the $\{|\nearrow\rangle, |\searrow\rangle\}$ basis. See Fig. III.5.

¶12. To find the result of applying filter B to the horizontally polarized light, we must express $|\rightarrow\rangle$ in the diagonal basis:

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\nearrow\rangle + |\searrow\rangle).$$

¶13. So if filter B $= \langle\nearrow|$ we get $|\nearrow\rangle$ with probability 1/2.

¶14. The effect of filter C, then, is to measure $|\nearrow\rangle$ by projecting against $\langle\uparrow|$. Note

$$|\nearrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\rightarrow\rangle).$$

¶15. Therefore we get $|\uparrow\rangle$ with another 1/2 decrease in intensity (so 1/8 overall).

B.6 No-cloning theorem

¶1. The *No-cloning Theorem* states that it is impossible to copy the state of a qubit.

¶2. On the contrary, assume that we have a unitary transformation U that does the copying, so that $U(|\psi\rangle \otimes |c\rangle) = |\psi\rangle \otimes |\psi\rangle$, where $|c\rangle$ is an arbitrary constant qubit.
That is $U|\psi c\rangle = |\psi\psi\rangle$.

¶3. Suppose $|\psi\rangle = a|0\rangle + b|1\rangle$.

¶4. By the linearity of U:

$$\begin{aligned}
U\ |\psi\rangle|c\rangle &= U\ (a|0\rangle + b|1\rangle)|c\rangle \\
&= U(a|0\rangle|c\rangle + b|1\rangle|c\rangle) \quad \text{distrib. of tensor prod.} \\
&= U(a|0c\rangle + b|1c\rangle) \\
&= a(U|0c\rangle) + b(U|1c\rangle) \quad \text{linearity} \\
&= a|00\rangle + b|11\rangle \quad \text{copying property.}
\end{aligned}$$

¶5. By expanding $|\psi\psi\rangle$ we have:

$$\begin{aligned}
U\ |\psi c\rangle &= |\psi\psi\rangle \\
&= (a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle) \\
&= a^2|00\rangle + ba|10\rangle + ab|01\rangle + b^2|11\rangle.
\end{aligned}$$

¶6. Note that these two expansions cannot be made equal in general, so no such unitary transformation exists.

¶7. Cloning is possible in the special cases $a = 0, b = 1$ or $a = 1, b = 0$, that is, where we know that we a cloning a pure basis state.

B.7 Entanglement

B.7.a Entangled and decomposable states

¶1. Suppose that $\mathcal{H}'$ and $\mathcal{H}''$ are the state spaces of two systems. Then $\mathcal{H} = \mathcal{H}' \otimes \mathcal{H}''$ is the state space of the *composite system*.

¶2. For simplicity, suppose that both spaces have the basis $\{|0\rangle, |1\rangle\}$. Then $\mathcal{H}' \otimes \mathcal{H}''$ has basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.
Recall that $|01\rangle = |0\rangle \otimes |1\rangle$, etc.

¶3. Arbitrary elements of $\mathcal{H}' \otimes \mathcal{H}''$ can be written in the form

$$\sum_{j,k=0,1} c_{jk}|jk\rangle = \sum_{j,k=0,1} c_{jk}\ |j'\rangle \otimes |k''\rangle.$$

¶4. Sometimes the state of the composite systems can be written as the tensor product of the states of the subsystems, $|\psi\rangle = |\psi'\rangle \otimes |\psi''\rangle$. Such a state is called a *separable*, *decomposable* or *product state*.

¶5. In other cases the state cannot be decomposed, in which case it is called an *entangled state*

¶6. **Bell entangled state:** For an example of an entangled state, consider the *Bell state* Φ^+, which might arise from a process that produced two particles with opposite spin (but without determining which is which):

$$\beta_{01} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \stackrel{\text{def}}{=} \Phi^+. \tag{III.4}$$

(The notations β_{01} and Φ^+ are both used.)
Note that the states $|01\rangle$ and $|10\rangle$ both have probability 1/2.

¶7. Such a state might arise from a process that emits two particles with opposite spin angular momentum in order to preserve conservation of spin angular momentum.

¶8. To show that it's entangled, we need to show that it cannot be decomposed, that is, that we cannot write $\beta_{01} = |\psi'\rangle \otimes |\psi''\rangle$, where $|\psi'\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\psi''\rangle = b_0|0\rangle + b_1|1\rangle$.

$$\beta_{01} \stackrel{?}{=} (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle).$$

Multiplying out the RHS yields:

$$a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle.$$

Therefore we must have $a_0b_0 = 0$ and $a_1b_1 = 0$. But this implies that either $a_0b_1 = 0$ or $a_1b_0 = 0$ (as opposed to $1/\sqrt{2}$), so the decomposition is impossible.

¶9. **Decomposable state:** Consider: $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Writing out the product $(a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$ as before, we require $a_0b_0 = a_0b_1 = a_1b_0 = a_1b_1 = \frac{1}{2}$. This is satisfied by $a_0 = a_1 = b_0 = b_1 = \frac{1}{\sqrt{2}}$.

¶10. **Bell states:** In addition to Eq. III.4, the other three Bell states are defined:

$$\beta_{00} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \stackrel{\text{def}}{=} \Psi^+, \tag{III.5}$$

$$\beta_{10} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \stackrel{\text{def}}{=} \Psi^-, \tag{III.6}$$

$$\beta_{11} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \stackrel{\text{def}}{=} \Phi^-. \tag{III.7}$$

¶11. The Ψ states have two identical qubits, the Φ states have opposite. The + superscript indicates they are added, the − that they are subtracted.

¶12. The general definition is:

$$\beta_{xy} = \frac{1}{\sqrt{2}}(|0, y\rangle + (-1)^x|1, \neg y\rangle).$$

B.7.b EPR paradox

¶1. Proposed by Einstein, Podolsky, and Rosen in 1935 to show problems in QM.

¶2. Suppose a source produces an entangled *EPR pair* (or *Bell state*) $\Psi^+ = \beta_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and the particles are sent to Alice and Bob.

¶3. If Alice measures her particle and gets $|0\rangle$, then that collapses the state to $|00\rangle$, and so Bob will have to get $|0\rangle$ if he measures. And likewise if Alice happens to get $|1\rangle$.

¶4. This happens instantaneously (but it does not permit faster-than-light communication).

¶5. **Hidden-variable theories:** One explanation is that there is some internal state in the particles that will determine the result of the measurement. Both particles have the same internal state.
This cannot explain the results of measurements in different bases.
In 1964 John Bell showed that any local hidden variable theory would lead to measurements satisfying a certain inequality (Bell's inequality).
Actual experiments violate Bell's inequality.
It has been verified over tens of kilometers.
Thus local hidden variable theories cannot be correct.

¶6. **Causal theories:** Another explanation is that Alice's measurement affects Bob's (or vice versa, if Bob measures first).
According to relativity theory, in some frames of reference Alice's measurement comes first, and in others, Bob's.
Therefore there is no consistent cause-effect relation.
This is why Alice and Bob cannot use entangled pairs to communicate.