Figure III.18: Superdense coding. [from ICQ]

C.6   Applications

C.6.a   SUPERDENSE CODING

¶1. **Superdense coding:** Also known as *dense coding*. A means by which one quantum particle can be used to transmit two classical bits of information.
(in general "only one classical bit's worth of information can be extracted from a qubit." [ICQ])

¶2. It was described by Bennett and Wiesner in 1992. It was partially validated experimentally in 1998.

¶3. Alice and Bob share an entangled pair.
To transmit two bits, Alice applies one of four transformations to her qubit.
She sends her qubit to Bob, who can apply an operation to the entangled pair to determine which of the four transformations she applied, and hence the two bits.

¶4. Alice and Bob share the entangled pair $\beta_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

¶5. To encode her two classical bits, Alice applies one of the four Pauli matrices (Sec. C.2.a, §6, p. 124), $\sigma_0, \sigma_1, \sigma_2, \sigma_3 = I, X, Y, Z$.

¶6. Alice applies this transformation to her qubit. Since Bob's bit is unaffected, the transformation on the entangled pair is $I \otimes I$, $X \otimes I$, $Y \otimes I$,

$Z \otimes I$.   The results are:

bits	transformation	result	
00	$I \otimes I$	$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$	$= \beta_{00}$
01	$X \otimes I$	$\frac{1}{\sqrt{2}}(\lvert 10\rangle + \lvert 01\rangle)$	$= \beta_{01}$
10	$Y \otimes I$	$\frac{1}{\sqrt{2}}(-\lvert 10\rangle + \lvert 01\rangle)$	$= \beta_{11}$
11	$Z \otimes I$	$\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$	$= \beta_{10}$

For example, in the last case, since $Z\lvert 0\rangle = \lvert 0\rangle$ and $Z\lvert 1\rangle = -\lvert 1\rangle$, we see $Z \otimes I \left[\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle) \right] = \frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$. Make sure you can explain the result in the other cases (Exer. III.31).

¶7. Alice sends her single transformed qubit to Bob.

¶8. Bob applies the CNOT gate to the transformed pair, which disentangles the qubits. Note that the result is decomposable:

bits	CNOT output	factored	
00	$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 10\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$	$\lvert 0\rangle$
01	$\frac{1}{\sqrt{2}}(\lvert 11\rangle + \lvert 01\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 1\rangle + \lvert 0\rangle)$	$\lvert 1\rangle$
10	$\frac{1}{\sqrt{2}}(-\lvert 11\rangle + \lvert 01\rangle)$	$\frac{1}{\sqrt{2}}(-\lvert 1\rangle + \lvert 0\rangle)$	$\lvert 1\rangle$
11	$\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 10\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)$	$\lvert 0\rangle$

For example, in the first case

$$\text{CNOT}\left[\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle) \right] = \frac{1}{\sqrt{2}}(\text{CNOT}\lvert 00\rangle + \text{CNOT}\lvert 11\rangle) = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 10\rangle).$$

Make sure you can explain the result in the other cases (Exer. III.32).

¶9. Since the state is decomposable, Bob can measure the second qubit without disturbing the first.

¶10. Notice that the first qubit is $\lvert +\rangle$ or $\lvert -\rangle$.

¶11. Finally, Bob applies the Hadamard gate to the first qubit:

bits	first qubit	result of H
00	$\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$	$\lvert 0\rangle$
01	$\frac{1}{\sqrt{2}}(\lvert 1\rangle + \lvert 0\rangle)$	$\lvert 0\rangle$
10	$\frac{1}{\sqrt{2}}(-\lvert 1\rangle + \lvert 0\rangle)$	$\lvert 1\rangle$
11	$\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)$	$\lvert 1\rangle$

For example, in the first case $\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle) = \lvert +\rangle$ and we know $H\lvert +\rangle = \lvert 0\rangle$. Make sure you can explain the result in the other cases (exercise).

¶12. All four input situations can be discriminated.

¶13. **Alternative approach:** The Bell states are orthonormal, and can be used to represent the two bits.

¶14. Recall the circuit for generating Bell states (Fig. III.13).
Its effect is $\mathrm{CNOT}(H \otimes I)\lvert xy\rangle = \lvert \beta_{xy}\rangle$.
This cannot be used by Alice for generating Bell states, because she doesn't have access to Bob's qubit.

¶15. However, Alice can use xy to select I, X, Z, or ZX (corresponding to $xy = 00, 01, 10, 11$ respectively) and generate the corresponding Bell state β_{xy}.
(Note that this is a different encoding from §6 above.)

¶16. **Measuring in Bell basis:** After Alice sends the qubit to Bob, he needs to measure it in the Bell basis.
This can be done by inverting the Bell state generator, which, since the CNOT and H are self-adjoint, is simply:

$$(H \otimes I)\mathrm{CNOT}\lvert \beta_{xy}\rangle = \lvert xy\rangle.$$

This translates the Bell basis into the computational basis, so Bob can measure the bits exactly.
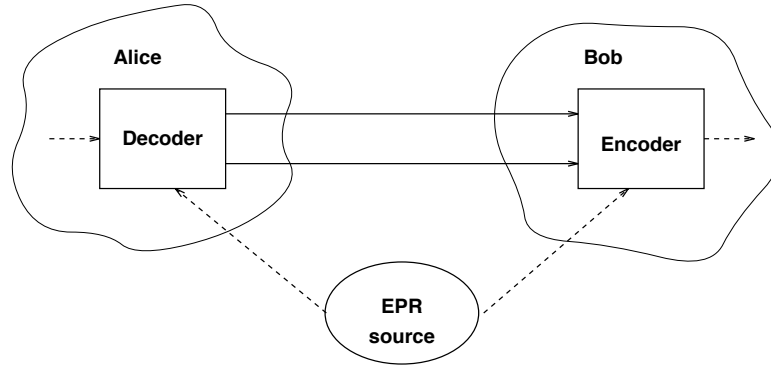
Figure III.19: Quantum teleportation. [fig. from ICQ]

C.6.b Quantum teleportation

¶1. The goal of quantum teleportation is to transfer the exact quantum state of a particle from Alice to Bob through a classical channel (Figs. III.19, III.20).

¶2. Of course, the no-cloning theorem says we cannot copy it, but we can teleport it by destroying the original. Furthermore, if Alice measures it, she will alter its state.

¶3. Single-bit quantum teleportation was described by Bennett in 1993 and first demonstrated experimentally in the late 1990s.

¶4. Alice and Bob begin by sharing the halves of an entangled pair, $\beta_{00} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

¶5. Suppose that the quantum state that Alice wants to share is $|\psi\rangle = a|0\rangle + b|1\rangle$.

¶6. The composite system comprising the unknown state and the Bell state is

$$
\begin{aligned}
|\psi, \beta_{00}\rangle &= (a|0\rangle + b|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}[a|0\rangle(|00\rangle + |11\rangle) + b|1\rangle(|00\rangle + |11\rangle)]
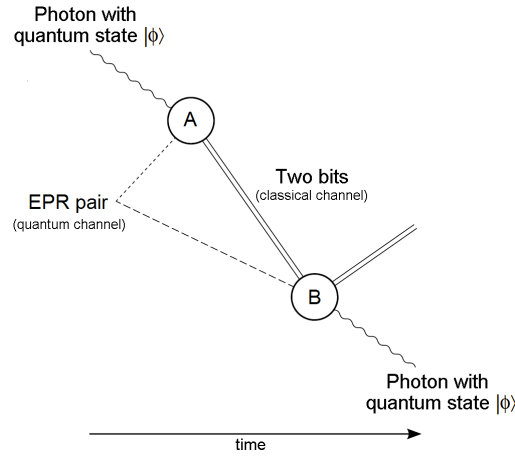\end{aligned}
$$

Figure III.20: Possible setup for quantum teleportation. [from wikipedia commons]

$$= \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle).$$

¶7. Alice applies the decoding circuit used for superdense coding to the unknown state and her qubit from the entangled pair. The function is $(H \otimes I \otimes I)(\text{CNOT} \otimes I)$.
This is the first step of measuring her two qubits in the Bell basis.

¶8. When Alice applies $\text{CNOT} \otimes I$ the result is:

$$(\text{CNOT} \otimes I)\left[\frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)\right]$$
$$= \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle).$$

¶9. When she applies $(H \otimes I \otimes I)$ the result is:

$$(H \otimes I \otimes I)\frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle)$$
$$= \frac{1}{2}\left[a(|000\rangle + |100\rangle + |011\rangle + |111\rangle) + b(|010\rangle - |110\rangle + |001\rangle - |101\rangle)\right].$$

This is because $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

¶10. Rearranging and factoring:

$$\frac{1}{2}\left[|00\rangle(a|0\rangle + b|1\rangle) + |01\rangle(a|1\rangle + b|0\rangle) + |10\rangle(a|0\rangle - b|1\rangle) + |11\rangle(a|1\rangle - b|0\rangle)\right].$$

¶11. Thus the amplitudes have been transferred from the first qubit to the third (Bob's), which now incorporates the amplitudes a and b, but in different ways depending on the first two bits. In fact you can see that they are transformed by the Pauli matrices.

¶12. Therefore Alice measures the first two bits (completing measurement in the Bell basis) and sends them to Bob over the classical channel. This measurement collapses the state, including Bob's qubit, but in a way that is determined by the first two qubits.

¶13. When Bob receives the two classical bits from Alice, he uses them to select a transformation for his qubit, which restores the amplitudes to the correct basis vectors. These transformations are the Pauli matrices:

bits	gate	input	
00	I	$a|0\rangle + b|1\rangle$	(identity)
01	X	$a|1\rangle + b|0\rangle$	(exchange)
10	Z	$a|0\rangle - b|1\rangle$	(negate)
11	ZX	$a|1\rangle - b|0\rangle$	(exchange–negate)

¶14. In each case, applying the specified gate to its input yields $|\psi\rangle = a|0\rangle + b|1\rangle$, Alice's original qubit. This is obvious in the 00 case, but you should verfy the others (exercise).

¶15. Notice that since Alice had to measure her original qubit, its state has collapsed.

¶16. Notice that quantum teleportation does not allow faster-than-light communication, since Alice had to transmit her two classical bits to Bob.

¶17. **Teleportation circuit:** The circuit in Fig. III.21 is slightly different, since it uses the fact that the appropriate transformations can be expressed in the form $Z^{M_1} X^{M_2}$, where M_1 and M_2 are the two classical bits. You should verify that $ZX = Y$ (exercise).
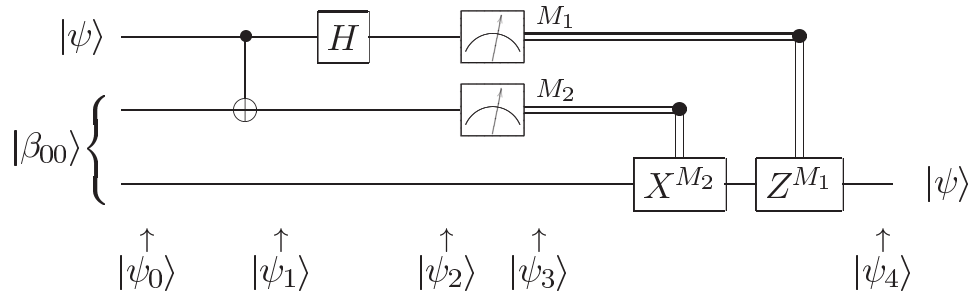
Figure III.21: Circuit for quantum teleportation. [from NC]

¶18. **Entangled states:** can be teleported.

¶19. **Interchanging resources:** Both superdense coding and teleportation indicate that under some circumstances two bits and an entangled pair can be interchanged with one qubit. This is one example of a method of *interchanging resources.*

¶20. **State of the art:** Free-space quantum teleportation has been demonstrated over 143 km between two of the Canary Islands (*Nature*, 13 Sept. 2012).[4]

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[4]`http://www.nature.com/nature/journal/v489/n7415/full/nature11472.html` (accessed 12-09-18).