C.7 Universal quantum gates

This lecture follows Nielsen & Chuang (2010, §4.5).

¶1. **Classical logic circuits:** Both the Fredkin (controlled swap) and Toffoli (controlled-controlled-NOT) gates are sufficiant for classical logic circuits.

¶2. But note that they can operate on qubits in superposition.

¶3. **Single-qubit unitary operators:** Single-qubit unitary operators can be approximated arbitrarily closely by the Hadamard and T ($\pi/8$) gates.

¶4. **$\pi/8$ or T gate:** The T or $\pi/8$ gate is defined:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \cong \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} \tag{III.19}$$

(ignoring global phase).

¶5. For an m-gate circuit and an accuracy of ϵ, $\mathcal{O}(m \log^c(m/\epsilon))$, where $c \approx 2$, gates are needed (Solovay-Kitaev theorem).

¶6. **Two-level unitary operations:** A *two-level operation* is one on a d-dimensional Hilbert space that non-trivially affects only two qubits out of n (where $d = 2^n$).

¶7. Any two-level unitary operation can be computed by a combination of CNOTs and single-qubit operations.

¶8. This requires $\mathcal{O}(n^2)$ single-qubit and CNOT gates.

¶9. **Arbitrary unitary matrix:** An arbitrary d-dimensional unitary matrix can be decomposed into a product of two-level unitary matrices.

¶10. At most $d(d-1)/2$ are required.
Therefore an operator on an n-qubit system requires at most $2^{n-1}(2^n - 1)$ two-level matrices.

¶11. **Conclusions:** The H (Hadamard), CNOT, and $\pi/8$ gates are sufficient.

¶12. **Fault-tolerance:** For fault-tolerance, either the *standard set* H (Hadamard), CNOT, $\pi/8$, and S (phase) can be used, or H, CNOT, Toffoli, and S.

¶13. S **or phase gate:** The *phase gate* is defined:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{III.20}$$

Note $S = T^2$.