D.5 Quantum error correction

This lecture follows IQC.

D.5.a Motivation

¶1. Quantum coherence is very difficult to maintain for long.
Even weak interactions with the environment can affect the quantum state, and we've seen that the amplitudes of the quantum state are critical to quantum algorithms.

¶2. On classical computers, bits are represented by very large numbers of particles (but that is changing).
On quantum computers, qubits are represented by atomic-scale states or objects (photons, nuclear spins, electrons, trapped ions, etc.)
They are very likely to become entangled with computationally irrelevant states of the computer and its environment, which are out of our control.

¶3. Quantum error correction is similar to classical error correction in that additional bits are introduced, creating redundancy that can be used to correct errors.

¶4. (a) It is different from classical error correction in that we want to restore the entire quantum state (i.e., the continuous amplitudes), not just 0s and 1s. Further, errors are continuous and can accumulate.
(b) Also, it must obey the no-cloning theorem.
(c) And measurement destroys quantum information.

D.5.b Effect of decoherence

¶1. Ideally the environment $|\Omega\rangle$, considered as a quantum system, does not interact with the computational state.
But if it does, the effect can be categorized as a unitary transformation on the environment-qubit system. Consider a single qubit:

$$D : \begin{cases} |\Omega\rangle|0\rangle & \Longrightarrow & |\Omega_{00}\rangle|0\rangle + |\Omega_{10}\rangle|1\rangle \\ |\Omega\rangle|1\rangle & \Longrightarrow & |\Omega_{01}\rangle|0\rangle + |\Omega_{11}\rangle|1\rangle \end{cases} .$$

¶2. In the case of no error, $|\Omega_{00}\rangle = |\Omega_{11}\rangle = |\Omega\rangle$ and $|\Omega_{01}\rangle = |\Omega_{10}\rangle = \mathbf{0}$.

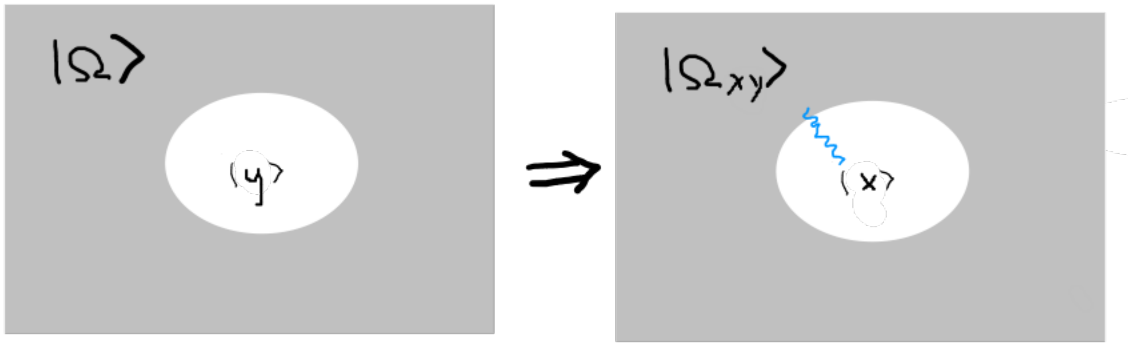

Figure III.34: Effects of decoherence on a qubit. On the left is a qubit $|y\rangle$ that is mostly isoloated from its environment $|\Omega\rangle$. On the right, a weak interaction between the qubit and the environment has led to a possibly altered qubit $|x\rangle$ and a correspondingly (slightly) altered environment $|\Omega_{xy}\rangle$.

¶3. If the entanglement is small, then $\|\Omega_{01}\|, \|\Omega_{10}\| \ll 1$.

¶4. Define *decoherence operators* $D_{xy}|\Omega\rangle = |\Omega_{xy}\rangle$, for $x, y \in \mathbf{2}$.

¶5. Then the evolution of the joint system is defined by the equations:

$$\begin{aligned} D|\Omega\rangle|0\rangle &= (D_{00} \otimes I + D_{10} \otimes X)|\Omega\rangle|0\rangle, \\ D|\Omega\rangle|1\rangle &= (D_{01} \otimes X + D_{11} \otimes I)|\Omega\rangle|1\rangle. \end{aligned}$$

¶6. Alternately, we can define it:

$$D = D_{00} \otimes |0\rangle\langle 0| + D_{10} \otimes |1\rangle\langle 0| + D_{01} \otimes |0\rangle\langle 1| + D_{11} \otimes |1\rangle\langle 1|.$$

¶7. It's easy to show (exercise):

$$|0\rangle\langle 0| = \frac{1}{2}(I+Z), |0\rangle\langle 1| = \frac{1}{2}(X-Y), |1\rangle\langle 0| = \frac{1}{2}(X+Y), |1\rangle\langle 1| = \frac{1}{2}(I-Z),$$

where $Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

¶8. Therefore

$$\begin{aligned} D &= \frac{1}{2}[D_{00} \otimes (I+Z) + D_{01} \otimes (X-Y) + \\ &\quad D_{10} \otimes (X+Y) + D_{11} \otimes (I-Z)] \\ &= \frac{1}{2}[(D_{00}+D_{11}) \otimes I + (D_{10}+D_{01}) \otimes X + \\ &\quad (D_{10}-D_{01}) \otimes Y + (D_{00}-D_{11}) \otimes Z]. \end{aligned}$$

¶9. Therefore the effect of decoherence on the qubit can be described by a linear combination of the Pauli matrices.
This is a distinctive feature about quantum errors: they have a finite basis, and because they are unitary, they are therefore invertible.

¶10. **Single qubits:** Single-qubit errors can be characterized in terms of a linear combination of the Pauli matrices (which span the space of 2×2 self-adjoint unitary matrices: ¶9, p. 120): I (no error), X (bit flip error), Y (phase error), and $Z = YX$ (bit flip phase error).

¶11. Therefore a single qubit error is represented by $e_0\sigma_0 + e_1\sigma_1 + e_2\sigma_2 + e_3\sigma_3 = \sum_{j=0}^{3} e_j\sigma_j$, where the σ_j are the Pauli matrices (¶6, p. 120).

D.5.c Correcting the quantum state

¶1. **Characterization of errors:** We consider a set of unitary "error operators" E_j, so that the error transformation is a superposition $E \stackrel{\text{def}}{=} \sum_j e_j E_j$.

¶2. **Quantum registers:** In the more general case of quantum registers, the E_j affect the entire register.

¶3. **Encoding:** An n-bit register is encoded in $n+m$ bits, where the extra bits are used for error correction.

¶4. Let $y = C(x)$ be the $n+m$ bit code for x.

¶5. Suppose $\tilde{y}$ is the result of error type k, $\tilde{y} = E_k(y)$.

¶6. **Syndrome:** Let $k = S(\tilde{y})$ be a function that determines the error syndrome, which identifies the error E_k from the corrupted code.

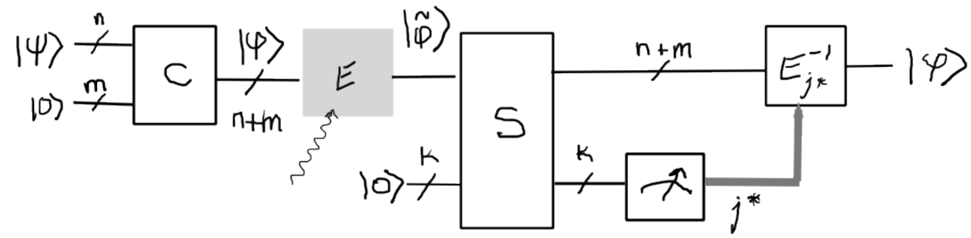

Figure III.35: Circuit for quantum error correction. $|\psi\rangle$ is the n-qubit quantum state to be encoded by C, which adds m error-correction qubits to yield the encoded state $|\phi\rangle$. E is a unitary superposition of error operators E_j, which alter the quantum state to $|\tilde{\phi}\rangle$. S is the syndrome extraction operator, which computes a superposition of codes for the errors E. The syndrome register is measured, to yield a particular syndrome code j^*, which is used to select a corresponding inverse error transformation $E^{-1}_{j^*}$ to correct the error.

¶7. **Correction:** Since the errors are unitary, and the syndrome is known, we can invert error and thereby correct it: $y = E^{-1}_{S(\tilde{y})}(\tilde{y})$.

¶8. **Quantum case:** Now consider the quantum case, in which the state $|\psi\rangle$ is a superposition of basis vectors, and the error is a superposition of error types, $E = \sum_j e_j E_j$.
See Fig. III.35.

¶9. **Encoding:** The encoded state is $|\phi\rangle \stackrel{\text{def}}{=} C|\psi\rangle$.

¶10. Let $|\tilde{\phi}\rangle = E|\phi\rangle$ be the code corrupted by error.

¶11. **Syndrome extraction:** Apply the syndrome extraction operator to the encoded state, augemented with enough qubits to represent the set of syndromes. This yields a superposition of syndromes:

$$S|\tilde{\phi}, \mathbf{0}\rangle = S\left(\sum_j e_j E_j|\phi\rangle\right)\otimes|\mathbf{0}\rangle = \sum_j e_j(SE_j|\phi\rangle|\mathbf{0}\rangle) = \sum_j e_j(E_j|\phi\rangle|j\rangle).$$

¶12. **Measurement:** Measure the syndrome register to obtain some j^* and the collapsed state $E_{j^*}|\phi\rangle|j^*\rangle$.

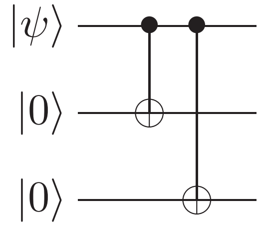

Figure III.36: Quantum encoding circuit for triple repetition code. [source: NC]

¶13. **Correction:** Apply $E_{j^*}^{-1}$ to correct the error.

¶14. Note that although there was a superposition of errors, we only have to correct one of them to get the original state back. This is because measurement collapses into a state affected by just that one error.

D.5.d Example

¶1. **Encoding:** For an example, suppose we use simple triple redundancy,

$$C|0\rangle = |000\rangle, \quad C|1\rangle = |111\rangle.$$

This is not a sophisticated code! It's called a *repetition code.*
The three-qubit codes are called *logical zero* and *logical one.*
See Fig. III.36

¶2. It can correct single bit flips (*by majority voting*), which are represented by the operators:

$$\begin{aligned} E_0 &= I \otimes I \otimes I \\ E_1 &= I \otimes I \otimes X \\ E_2 &= I \otimes X \otimes I \\ E_3 &= X \otimes I \otimes I. \end{aligned}$$

¶3. **Syndrome:** The following works as a syndrome extraction operator:

$$S|x_3, x_2, x_1, 0, 0, 0\rangle \stackrel{\text{def}}{=} |x_3, x_2, x_1, x_1 \oplus x_2, x_1 \oplus x_3, x_2 \oplus x_3\rangle.$$

The $\oplus$s compare each pair of bits, and so the $\oplus$ will be zero if the two bits are the same (the majority).

¶4. **Correction:** The following table shows the bit flipped (if any), the correponding syndrome, and the operator to correct it (which is the same as the operator that caused the error):

bit flipped	syndrome	error correction
none	$\|000\rangle$	$I \otimes I \otimes I$
1	$\|110\rangle$	$I \otimes I \otimes X$
2	$\|101\rangle$	$I \otimes X \otimes I$
3	$\|011\rangle$	$X \otimes I \otimes I$

¶5. **Example state:** Suppose we want to encode the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

¶6. **Code:** Its code is $|\phi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$.

¶7. **Error:** Suppose the following error occurs: $E = \frac{4}{5} X \otimes I \otimes I + \frac{3}{5} I \otimes X \otimes I$ (that is, the bit 3 flips with probability 16/25, and bit 2 with probability 9/25).

¶8. **Error state:** The resulting error state is

$$\begin{aligned}
|\tilde{\phi}\rangle &= E|\phi\rangle \\
&= \left(\frac{4}{5} X \otimes I \otimes I + \frac{3}{5} I \otimes X \otimes I\right) \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle) \\
&= \frac{4}{5\sqrt{2}} X \otimes I \otimes I(|000\rangle - |111\rangle) + \frac{3}{5\sqrt{2}} I \otimes X \otimes I(|000\rangle - |111\rangle) \\
&= \frac{4}{5\sqrt{2}}(|100\rangle - |011\rangle) + \frac{3}{5\sqrt{2}}(|010\rangle - |101\rangle).
\end{aligned}$$

¶9. **Syndrome:** Applying the syndrome extraction operator yields:

$$\begin{aligned}
S|\tilde{\phi}, 000\rangle &= S\left[\frac{4}{5\sqrt{2}}(|100000\rangle - |011000\rangle) + \frac{3}{5\sqrt{2}}(|010000\rangle - |101000\rangle)\right] \\
&= \frac{4}{5\sqrt{2}}(|100011\rangle - |011011\rangle) + \frac{3}{5\sqrt{2}}(|010101\rangle - |101101\rangle) \\
&= \frac{4}{5\sqrt{2}}(|100\rangle - |011\rangle) \otimes |011\rangle + \frac{3}{5\sqrt{2}}(|010\rangle - |101\rangle) \otimes |101\rangle
\end{aligned}$$

¶10. **Measurement:** Measuring the syndrome register yields either $|011\rangle$ (representing an error in bit 3) or $|101\rangle$ (representing an error in bit 2). Suppose we get $|011\rangle$. The state collapses into:

$$\frac{1}{\sqrt{2}}(|100\rangle - |011\rangle) \otimes |011\rangle.$$

Note that we have projected into a subspace for just one of the two bit-flip errors that occurred (the flip in bit 3).

¶11. **Correction:** The measured syndrome $|011\rangle$ tells us to apply $X \otimes I \otimes I$ to the first three bits, which restores $|\phi\rangle$:

$$(X \otimes I \otimes I)\frac{1}{\sqrt{2}}(|100\rangle - |011\rangle) = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle) = |\phi\rangle.$$

D.5.e Discussion

¶1. **Shor code:** There is a nine-qubit code, called the *Shor code*, that can correct arbitrary errors on a single qubit, even replacing the entire qubit by garbage.

¶2. An entire continuum of errors can be corrected by correcting only a discrete set of errors.
This works in quantum computation, but not classical analog computing.

¶3. **Fault-tolerant quantum computation:** What do we do about noise in the gates that do encoding and decoding?
It is possible to do *fault-tolerant quantum computation*.
"Even more impressively, fault-tolerance allow us to perform logical operations on encoded quantum states, in a manner which tolerates faults in the underlying gate operations." [NC 425]

¶4. **Threshold theorem:** "provided the noise in individual quantum gates is below a certain constant threshold it is possible to efficiently perform an arbitrarily large quantum computation." [NC 425][10]

[10]See NC §10.6.4.