

## J Exercises

**Exercise III.1** Prove that projectors are idempotent, that is,  $P^2 = P$ .

**Exercise III.2** Prove that a normal matrix is Hermitian iff it has real eigenvalues.

**Exercise III.3** Prove that  $U(t) \stackrel{\text{def}}{=} \exp(-iHt/\hbar)$  is unitary.

**Exercise III.4** Use spectral decomposition to show that  $K = -i \log(U)$  is Hermitian for any unitary  $U$ , and thus  $U = \exp(iK)$  for some Hermitian  $K$ .

**Exercise III.5** Show that  $[L, M]$  and  $\{L, M\}$  are bilinear operators (linear in both of their arguments).

**Exercise III.6** Show that  $[L, M]$  is anticommutative, i.e.,  $[M, L] = -[L, M]$ , and that  $\{L, M\}$  is commutative.

**Exercise III.7** Show that  $LM = \frac{[L, M] + \{L, M\}}{2}$ .

**Exercise III.8** Show that the four Bell states are orthonormal.

**Exercise III.9** What is the effect of  $Y$  (imaginary definition) on the computational basis vectors? What is its effect if you use the real definition (¶12, p. 121)?

**Exercise III.10** Prove that  $I, X, Y$ , and  $Z$  are unitary. Use either the imaginary or real definition of  $Y$  (¶12, p. 121).

**Exercise III.11** Show that the  $X, Y, Z$  and  $H$  gates are Hermitian (their own inverses) and prove your answers. Use either the imaginary or real definition of  $Y$  (¶12, p. 121).

**Exercise III.12** Prove the following useful identities:

$$HXH = Z, HYH = -Y, HZH = X.$$

**Exercise III.13** Show (using the real definition of  $Y$ , ¶12, p. 121):  
 $|0\rangle\langle 0| = \frac{1}{2}(I + Z)$ ,  $|0\rangle\langle 1| = \frac{1}{2}(X - Y)$ ,  $|1\rangle\langle 0| = \frac{1}{2}(X + Y)$ ,  $|1\rangle\langle 1| = \frac{1}{2}(I - Z)$ .

**Exercise III.14** Prove  $|\beta_{xy}\rangle = (P \otimes I)|\beta_{00}\rangle$ , where  $xy = 00, 01, 11, 10$  for  $P = I, X, Y, Z$ , respectively.

**Exercise III.15** Suppose that  $P$  is one of the Pauli operators, but you don't know which one. However, you are able to pick a 2-qubit state  $|\psi_0\rangle$  and operate on it,  $|\psi_1\rangle = (P \otimes I)|\psi_0\rangle$ . Further, you are able to select a unitary operation  $U$  to apply to  $|\psi_1\rangle$ , and to measure the 2-qubit result,  $|\psi_2\rangle = U|\psi_1\rangle$ , in the computational basis. Select  $|\psi_0\rangle$  and  $U$  so that you can determine with certainty the unknown Pauli operator  $P$ .

**Exercise III.16** What is the matrix for CNOT in the standard basis? Prove your answer.

**Exercise III.17** Show that CNOT does not violate the No-cloning Theorem by showing that, in general,  $\text{CNOT}|\psi\rangle|0\rangle \neq |\psi\rangle|\psi\rangle$ . Under what conditions does the equality hold?

**Exercise III.18** What is the matrix for CCNOT in the standard basis? Prove your answer.

**Exercise III.19** Use a single Toffoli gate to implement each of NOT, NAND, and XOR.

**Exercise III.20** Use Toffoli gates to implement FAN-OUT. FAN-OUT would seem to violate the No-cloning Theorem, but it doesn't. Explain why.

**Exercise III.21** Show that  $|+\rangle, |-\rangle$  is an ON basis.

**Exercise III.22** Prove:

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \\ |1\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \end{aligned}$$

**Exercise III.23** Prove that  $Z|+\rangle = |-\rangle$  and  $Z|-\rangle = |+\rangle$ .

**Exercise III.24** Prove:

$$\begin{aligned} H(a|0\rangle + b|1\rangle) &= a|+\rangle + b|-\rangle, \\ H(a|+\rangle + b|-\rangle) &= a|0\rangle + b|1\rangle. \end{aligned}$$

**Exercise III.25** Prove  $H = (X + Z)/\sqrt{2}$ .

**Exercise III.26** Prove Eq. III.18 (p. 128).

**Exercise III.27** Show that three successive CNOTs, connected as in Fig. III.11 (p. 126), will swap two qubits.

**Exercise III.28** Recall the conditional selection between two operators (¶14, p. 127):  $|0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1$ . Suppose the control bit is a superposition  $|\chi\rangle = a|0\rangle + b|1\rangle$ . Show that:

$$(|0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1)|\chi, \psi\rangle = a|0, U_0\psi\rangle + b|1, U_1\psi\rangle.$$

**Exercise III.29** Show that the 1-bit full adder (Fig. III.15, p. 129) is correct.

**Exercise III.30** Show that the operator  $U_f$  is unitary:

$$U_f|x, y\rangle \stackrel{\text{def}}{=} |x, y \oplus f(x)\rangle,$$

**Exercise III.31** Verify the remaining superdense encoding transformations in Sec. C.6.a, ¶5 (p. 133).

**Exercise III.32** Verify the remaining superdense decoding transformations in Sec. C.6.a, ¶8 (p. 134).

**Exercise III.33** Complete the following step from the derivation of the Deutsch-Jozsa algorithm (Sec. D.1.b, ¶11, p. 146):

$$H|x\rangle = \sum_{z \in \mathbf{2}} \frac{1}{\sqrt{2}} (-1)^{xz} |z\rangle.$$

**Exercise III.34** Show that  $\text{CNOT}(H \otimes I) = (I \otimes H)C_Z H^{\otimes 2}$ , where  $C_Z$  is the controlled- $Z$  gate.

**Exercise III.35** Show that the Fourier transform matrix (¶11, p. 153, Sec. D.3.a) is unitary.

**Exercise III.36** Prove the claim in ¶29, p. 170 (Sec. D.4.b).

**Exercise III.37** Prove the claim in ¶32, p. 170 (Sec. D.4.b).

**Exercise III.38** Design a quantum gate array for the following syndrome extraction operator (¶3, p. 179, in Sec. D.5.d, p. 179):

$$S|x_3, x_2, x_1, 0, 0, 0\rangle \stackrel{\text{def}}{=} |x_3, x_2, x_1, x_1 \oplus x_2, x_1 \oplus x_3, x_2 \oplus x_3\rangle.$$

**Exercise III.39** Design a quantum gate array to apply the appropriate error correction for the extracted syndrome as given in ¶4, p. 180 (Sec. D.5.d, p. 179):

| bit flipped | syndrome      | error correction        |
|-------------|---------------|-------------------------|
| none        | $ 000\rangle$ | $I \otimes I \otimes I$ |
| 1           | $ 110\rangle$ | $I \otimes I \otimes X$ |
| 2           | $ 101\rangle$ | $I \otimes X \otimes I$ |
| 3           | $ 011\rangle$ | $X \otimes I \otimes I$ |

**Exercise III.40** Prove that  $A_a A_a = \mathbf{1}$  (Sec. F.1.b).

**Exercise III.41** Prove that  $A_{ab,c} = \mathbf{1} + a^* a b^* b (c + c^* - \mathbf{1}) = \mathbf{1} + N_a N_b (A_c - \mathbf{1})$  is a correct definition of CCNOT by showing how it transforms the quantum register  $|a, b, c\rangle$  (Sec. F.1.b).

**Exercise III.42** Show that the following definition of Feynman's switch is unitary (Sec. F.1.b):

$$q^* c p + r^* c^* p + p^* c^* q + p^* c r.$$