B.5 Superposition

B.5.a Bases

¶1. In QM certain physical quantities are quantized, such as the energy of an electron in an atom.
Therefore an atom might be in certain distinct energy states $|\text{ground}\rangle$, $|\text{first excited}\rangle$, $|\text{second excited}\rangle$, ...

¶2. Other particles might have distinct states such as spin-up $|\uparrow\rangle$ and spin-down $|\downarrow\rangle$.

¶3. In each case these alternative states are orthonormal: $\langle\uparrow|\downarrow\rangle = 0$; $\langle\text{ground} \mid \text{first excited}\rangle = 0$, $\langle\text{ground} \mid \text{second excited}\rangle = 0$, $\langle\text{first excited} \mid \text{second excited}\rangle = 0$.

¶4. In general we may express the same state with respect to different bases, such as vertical or horizontal polarization $|\rightarrow\rangle$, $|\uparrow\rangle$; or orthogonal diagonal polarizations $|\nearrow\rangle$, $|\searrow\rangle$.

B.5.b Superpositions of Basis States

¶1. One of the unique characteristics of QM is that a physical system can be in a superposition of basis states, for example,

$$|\psi\rangle = c_0|\text{ground}\rangle + c_1|\text{first excited}\rangle + c_2|\text{second excited}\rangle,$$

where the c_j are complex numbers, called *(probability) amplitudes*.

¶2. Since $\| |\psi\rangle \| = 1$, we know $|c_0|^2 + |c_1|^2 + |c_2|^2 = 1$.

¶3. With respect to a given basis, a state $|\psi\rangle$ is interchangeable with its vector of coefficients, $\mathbf{c} = (c_0, c_1, \ldots, c_n)^{\mathrm{T}}$. When the basis is understood, we can use $|\psi\rangle$ as a name for this vector.

¶4. **Quantum parallelism:** The ability of a quantum system to be in many states simultaneously is the foundation of *quantum parallelism*.

¶5. **Measurement:** As we will see, when we measure the quantum state

$$c_0|E_0\rangle + c_1|E_1\rangle + \ldots + c_n|E_n\rangle$$

with respect to the $|E_0\rangle, \ldots, |E_n\rangle$ basis, we will get the result $|E_j\rangle$ with probability $|c_j|^2$ and the state will "collapse" into state $|E_j\rangle$.

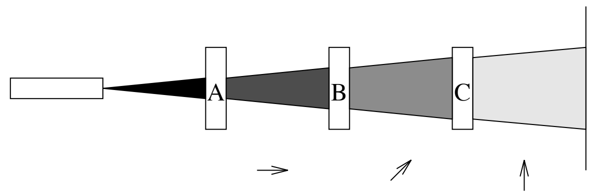

Figure III.4: Fig. from IQC.

¶6. **Qubit:** For the purposes of quantum computation, we usually pick two basis states and use them to represent the bits 1 and 0, for example, $|1\rangle = |\text{ground}\rangle$ and $|0\rangle = |\text{excited}\rangle$.
I've picked the opposite of the "obvious" assignment ($|0\rangle = |\text{ground}\rangle$) just to show that the assignment is arbitrary (just as for classical bits).

¶7. Note that $|0\rangle \neq \mathbf{0}$, the zero element of the vector space, since $\| |0\rangle \| = 1$ but $\|\mathbf{0}\| = 0$. (Thus $\mathbf{0}$ does not represent a physical state.)

B.5.c Photon polarization experiment

See Fig. III.4.

¶1. **Experiment:** Suppose we have three polarizing filters, A, B, and C, polarized horizontally, 45°, and vertically, respectively.

¶2. Place filter A between strong light source and screen. Intensity is reduced by half and light is horizontally polarized.
(Note: intensity would be much less if it allowed only horizontally polarized light through, as in sieve model.)

¶3. Insert filter C and intensity drops to zero. No surprise, since cross-polarized.

¶4. Insert filter B between A and C, and some light (about 1/8 intensity) will return!
Can't be explained by sieve model.

¶5. **Explanation:** A photon's polarization state can be represented by a unit vector pointing in appropriate direction.

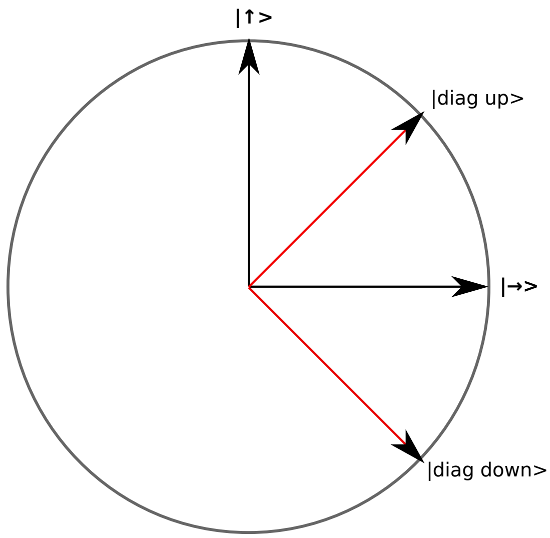

Figure III.5: Alternative polarization bases for measuring photons (black = rectilinear basis, red = diagonal basis). Note $|\nearrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\rightarrow\rangle)$ and $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\nearrow\rangle + |\searrow\rangle)$.

¶6. Arbitrary polarization can be expressed by $a|0\rangle + b|1\rangle$ for any two basis vectors $|0\rangle$, $|1\rangle$, where $|a|^2 + |b|^2 = 1$.

¶7. A polarizing filter measures a state with respect to a basis that includes a vector parallel to polarization and one orthogonal to it.

¶8. The filter A is the projector $|\rightarrow\rangle\langle\rightarrow|$.
To get the probability amplitude, apply to $|\psi\rangle \overset{\text{def}}{=} a|\rightarrow\rangle + b|\uparrow\rangle$:

$$\langle\rightarrow|\,\psi\rangle = \langle\rightarrow|(a|\rightarrow\rangle + b|\uparrow\rangle) = a\langle\rightarrow|\rightarrow\rangle + b\langle\rightarrow|\uparrow\rangle = a.$$

So with probability $|a|^2$ we get $|\rightarrow\rangle$. Recall (Eqn. III.1, p. 91):

$$p(|\rightarrow\rangle) = \|\langle\rightarrow|\,\psi\rangle\|^2 = |a|^2.$$

¶9. So if the polarizations are randomly distributed from the source, half will get through with resulting photons all $|\rightarrow\rangle$.
Why 1/2? Note $a = \cos\theta$ and $\langle a^2\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\, \mathrm{d}\theta = \frac{1}{2}$.

¶10. When we insert filter C we are measuring with $\langle\uparrow|$ and the result is 0, as expected.

¶11. **Diagonal filter:** Filter B measures with respect to the $\{|\nearrow\rangle, |\searrow\rangle\}$ basis. See Fig. III.5.

¶12. To find the result of applying filter B to the horizontally polarized light, we must express $|\rightarrow\rangle$ in the diagonal basis:

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\nearrow\rangle + |\searrow\rangle).$$

¶13. So if filter B $= \langle\nearrow|$ we get $|\nearrow\rangle$ with probability 1/2.

¶14. The effect of filter C, then, is to measure $|\nearrow\rangle$ by projecting against $\langle\uparrow|$. Note

$$|\nearrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\rightarrow\rangle).$$

¶15. Therefore we get $|\uparrow\rangle$ with another 1/2 decrease in intensity (so 1/8 overall).