B.6 No-cloning theorem

¶1. The *No-cloning Theorem* states that it is impossible to copy the state of a qubit.

¶2. On the contrary, assume that we have a unitary transformation U that does the copying, so that $U(|\psi\rangle \otimes |c\rangle) = |\psi\rangle \otimes |\psi\rangle$, where $|c\rangle$ is an arbitrary constant qubit.
That is $U|\psi c\rangle = |\psi\psi\rangle$.

¶3. Suppose $|\psi\rangle = a|0\rangle + b|1\rangle$.

¶4. By the linearity of U:

$$\begin{array}{rcll} U\ |\psi\rangle|c\rangle & = & U\ (a|0\rangle + b|1\rangle)|c\rangle & \\ & = & U(a|0\rangle|c\rangle + b|1\rangle|c\rangle) & \text{distrib. of tensor prod.} \\ & = & U(a|0c\rangle + b|1c\rangle) & \\ & = & a(U|0c\rangle) + b(U|1c\rangle) & \text{linearity} \\ & = & a|00\rangle + b|11\rangle & \text{copying property.} \end{array}$$

¶5. By expanding $|\psi\psi\rangle$ we have:

$$\begin{array}{rcl} U\ |\psi c\rangle & = & |\psi\psi\rangle \\ & = & (a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle) \\ & = & a^2|00\rangle + ba|10\rangle + ab|01\rangle + b^2|11\rangle. \end{array}$$

¶6. Note that these two expansions cannot be made equal in general, so no such unitary transformation exists.

¶7. Cloning is possible in the special cases $a = 0, b = 1$ or $a = 1, b = 0$, that is, where we know that we a cloning a pure basis state.

B.7 Entanglement

B.7.a ENTANGLED AND DECOMPOSABLE STATES

¶1. Suppose that $\mathcal{H}'$ and $\mathcal{H}''$ are the state spaces of two systems. Then $\mathcal{H} = \mathcal{H}' \otimes \mathcal{H}''$ is the state space of the *composite system*.

¶2. For simplicity, suppose that both spaces have the basis $\{|0\rangle, |1\rangle\}$. Then $\mathcal{H}' \otimes \mathcal{H}''$ has basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.
Recall that $|01\rangle = |0\rangle \otimes |1\rangle$, etc.

¶3. Arbitrary elements of $\mathcal{H}' \otimes \mathcal{H}''$ can be written in the form

$$\sum_{j,k=0,1} c_{jk}|jk\rangle = \sum_{j,k=0,1} c_{jk}\ |j'\rangle \otimes |k''\rangle.$$

¶4. Sometimes the state of the composite systems can be written as the tensor product of the states of the subsystems, $|\psi\rangle = |\psi'\rangle \otimes |\psi''\rangle$. Such a state is called a *separable*, *decomposable* or *product state*.

¶5. In other cases the state cannot be decomposed, in which case it is called an *entangled state*

¶6. **Bell entangled state:** For an example of an entangled state, consider the *Bell state* Φ^+, which might arise from a process that produced two particles with opposite spin (but without determining which is which):

$$\beta_{01} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \stackrel{\text{def}}{=} \Phi^+. \tag{III.4}$$

(The notations β_{01} and Φ^+ are both used.)
Note that the states $|01\rangle$ and $|10\rangle$ both have probability 1/2.

¶7. Such a state might arise from a process that emits two particles with opposite spin angular momentum in order to preserve conservation of spin angular momentum.

¶8. To show that it's entangled, we need to show that it cannot be decomposed, that is, that we cannot write $\beta_{01} = |\psi'\rangle \otimes |\psi''\rangle$, where $|\psi'\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\psi''\rangle = b_0|0\rangle + b_1|1\rangle$.

$$\beta_{01} \stackrel{?}{=} (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle).$$

Multiplying out the RHS yields:

$$a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle.$$

Therefore we must have $a_0 b_0 = 0$ and $a_1 b_1 = 0$. But this implies that either $a_0 b_1 = 0$ or $a_1 b_0 = 0$ (as opposed to $1/\sqrt{2}$), so the decomposition is impossible.

¶9. **Decomposable state:** Consider: $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Writing out the product $(a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$ as before, we require $a_0 b_0 = a_0 b_1 = a_1 b_0 = a_1 b_1 = \frac{1}{2}$. This is satisfied by $a_0 = a_1 = b_0 = b_1 = \frac{1}{\sqrt{2}}$.

¶10. **Bell states:** In addition to Eq. III.4, the other three Bell states are defined:

$$\beta_{00} \overset{\text{def}}{=} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \overset{\text{def}}{=} \Psi^+, \tag{III.5}$$

$$\beta_{10} \overset{\text{def}}{=} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \overset{\text{def}}{=} \Psi^-, \tag{III.6}$$

$$\beta_{11} \overset{\text{def}}{=} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \overset{\text{def}}{=} \Phi^-. \tag{III.7}$$

¶11. The Ψ states have two identical qubits, the Φ states have opposite. The + superscript indicates they are added, the − that they are subtracted.

¶12. The general definition is:

$$\beta_{xy} = \frac{1}{\sqrt{2}}(|0, y\rangle + (-1)^x |1, \neg y\rangle).$$

B.7.b EPR PARADOX

¶1. Proposed by Einstein, Podolsky, and Rosen in 1935 to show problems in QM.

¶2. Suppose a source produces an entangled *EPR pair* (or *Bell state*) $\Psi^+ = \beta_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and the particles are sent to Alice and Bob.

¶3. If Alice measures her particle and gets $|0\rangle$, then that collapses the state to $|00\rangle$, and so Bob will have to get $|0\rangle$ if he measures. And likewise if Alice happens to get $|1\rangle$.

¶4. This happens instantaneously (but it does not permit faster-than-light communication).

¶5. **Hidden-variable theories:** One explanation is that there is some internal state in the particles that will determine the result of the measurement. Both particles have the same internal state.
This cannot explain the results of measurements in different bases.
In 1964 John Bell showed that any local hidden variable theory would lead to measurements satisfying a certain inequality (Bell's inequality).
Actual experiments violate Bell's inequality.
It has been verified over tens of kilometers.
Thus local hidden variable theories cannot be correct.

¶6. **Causal theories:** Another explanation is that Alice's measurement affects Bob's (or vice versa, if Bob measures first).
According to relativity theory, in some frames of reference Alice's measurement comes first, and in others, Bob's.
Therefore there is no consistent cause-effect relation.
This is why Alice and Bob cannot use entangled pairs to communicate.