## C.7 Universal quantum gates

This lecture follows Nielsen & Chuang (2010, §4.5).

- **¶1.** Classical logic circuits: Both the Fredkin (controlled swap) and Toffoli (controlled-controlled-NOT) gates are sufficient for classical logic circuits.
- ¶2. But note that they can operate on qubits in superposition.
- ¶3. Single-qubit unitary operators: Single-qubit unitary operators can be approximated arbitrarily closely by the Hadamard and T ( $\pi/8$ ) gates.
- ¶4.  $\pi/8$  or T gate: The T or  $\pi/8$  gate is defined:

$$T = \begin{pmatrix} 1 & 0\\ 0 & e^{i\pi/4} \end{pmatrix} \cong \begin{pmatrix} e^{-i\pi/8} & 0\\ 0 & e^{i\pi/8} \end{pmatrix}$$
(III.19)

(ignoring global phase).

- ¶5. For an *m*-gate circuit and an accuracy of  $\epsilon$ ,  $\mathcal{O}(m \log^c(m/\epsilon))$ , where  $c \approx 2$ , gates are needed (Solovay-Kitaev theorem).
- ¶6. Two-level unitary operations: A two-level operation is one on a d-dimensional Hilbert space that non-trivially affects only two qubits out of n (where  $d = 2^n$ ).
- ¶7. Any two-level unitary operation can be computed by a combination of CNOTs and single-qubit operations.
- ¶8. This requires  $\mathcal{O}(n^2)$  single-qubit and CNOT gates.
- **¶**9. **Arbitrary unitary matrix:** An arbitrary *d*-dimensional unitary matrix can be decomposed into a product of two-level unitary matrices.
- ¶10. At most d(d-1)/2 are required. Therefore an operator on an *n*-qubit system requires at most  $2^{n-1}(2^n - 1)$  two-level matrices.
- ¶11. Conclusions: The H (Hadamard), CNOT, and  $\pi/8$  gates are sufficient.

## C. QUANTUM INFORMATION

- ¶12. Fault-tolerance: For fault-tolerance, either the standard set H (Hadamard), CNOT,  $\pi/8$ , and S (phase) can be used, or H, CNOT, Toffoli, and S.
- ¶13. S or phase gate: The phase gate is defined:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$
(III.20)

Note  $S = T^2$ .