

## F Universal quantum computers

- ¶1. **Power:** A natural question is: What is the power of a quantum computer?  
Is it super-Turing or sub-Turing?
- ¶2. **Efficiency:** Another question is: What is its efficiency?  
Can it solve NP problems efficiently?
- ¶3. **Models:** There are a number of universal QC models for both theoretical and practical purposes.

### F.1 Feynman on quantum computation

#### F.1.a SIMULATING QUANTUM SYSTEMS

This section is based primarily on F82.

- ¶1. In 1982 Richard Feynman discussed what would be required to simulate a quantum mechanical system on a digital computer.
- ¶2. **Probabilistic classical system:** First he considered a classical probabilistic physical system.  
Suppose we want to use a conventional computer to calculate the probabilities as the system evolves in time.
- ¶3. Suppose the system comprises  $R$  particles that are confined to  $N$  locations in space.  
Each configuration  $c$  has a probability  $p(c)$ .  
There are  $N^R$  possible configurations, since a configuration assigns a location  $N$  to each of the  $R$  particles (i.e., the number of functions  $R \rightarrow N$ ).
- ¶4. Therefore to simulate all the possibilities would require keeping track of a number of quantities (the probabilities) that grows exponentially with the size of the system.  
This is infeasible.
- ¶5. So let's take a weaker goal: we want a simulator that exhibits the same probabilistic behavior as the system.

So if we run both of them over and over, we will see the same distribution of behaviors.

This we can do.

- ¶6. You can implement this by having a nondeterministic computer that has the same state transition probabilities as the primary system.
- ¶7. **Quantum system:** Let's try the same trick with quantum systems, i.e., have a conventional computer that exhibits the same probabilities as the quantum system.
- ¶8. If you do the math (which we won't), it turns out that this is impossible. The reason is that, in effect, some of the state transitions would have to have what amount to negative probabilities, and we don't know how to do this classically.  
We've seen how in QM, probabilities can in effect cancel by destructive interference of the wavefunctions.
- ¶9. The conclusion is that no conventional computer can efficiently simulate a quantum computer.
- ¶10. Therefore, if we want to (efficiently) simulate any physical system, we need a quantum computer.

### F.1.b UNIVERSAL QUANTUM COMPUTER

This section is based primarily on F85.

- ¶1. In 1985 Feynman described several possible designs for a universal quantum computer.
- ¶2. He observes that NOT, CNOT, and CCNOT are sufficient for any logic gate, as well as for COPY and EXCHANGE, and therefore for universal computation.
- ¶3. **Adder:** He exhibits circuits for a simple adder (Fig. III.38) and a full adder (Fig. III.39).
- ¶4. **Hamiltonian:** The goal is to construct a Hamiltonian to govern the operation of a quantum computer.

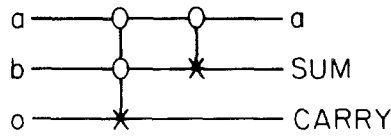


Figure III.38: Simple adder using reversible logic. [fig. from F85]

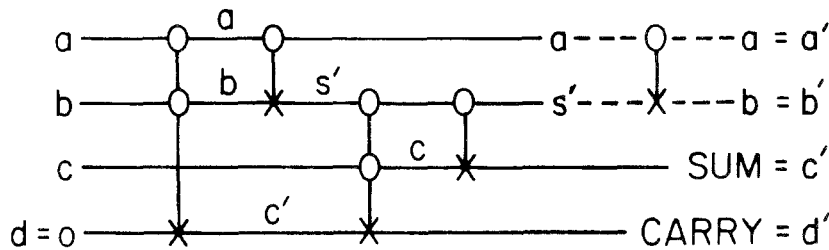


Figure III.39: Full adder using reversible logic. [fig. from F85]

- ¶5. **Primitive operations:** F describes quantum logic gates in terms of two primitive operations, which change the state of an “atom” (two-state system or “wire”).
- ¶6. Letters near the beginning of the alphabet ( $a, b, c, \dots$ ) are used for *data* or *register atoms*, and those toward the end ( $p, q, r, \dots$ ) for *program atoms* (which are used for sequencing operations).  
In this simple sequential computer, only one program atom is set at a time.
- ¶7. **Annihilation operator:** For a single line  $a$ , the *annihilation operator* is defined:

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 1|.$$

The *annihilator* changes the state  $|1\rangle$  to  $|0\rangle$ . Applied to  $|0\rangle$ , it leaves the state unchanged and returns the *zero vector*  $\mathbf{0}$  (which is not a meaningful quantum state).

It *matches*  $|1\rangle$  and *resets* it to  $|0\rangle$ .

It’s not unitary (because not norm preserving).

This is a “partial NOT” operation.

¶8. **Creation operation:** Its conjugate is the *creation operation*

$$a^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0|.$$

The *creator* transforms  $|0\rangle$  to  $|1\rangle$ , but leaves  $|1\rangle$  alone, returning  $\mathbf{0}$ . It matches  $|0\rangle$  and resets it to  $|1\rangle$ .

Note that  $a^*$  is the adjoint (conjugate transpose) of  $a$ .

This is the other half of NOT.

¶9. **Number operation or 1-test:** Consider<sup>11</sup>

$$N_a = a^*a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|.$$

This has the effect of returning  $|1\rangle$  for input  $|1\rangle$ , but  $\mathbf{0}$  for  $|0\rangle$ :

$$N_a = a^*a = |1\rangle\langle 0| |0\rangle\langle 1| + |1\rangle\langle 0| |0\rangle\langle 0| + |1\rangle\langle 1| |1\rangle\langle 1| = |1\rangle\langle 1|.$$

Thus it's a test for  $|1\rangle$ .

(This is a partial identity operation.)

¶10. **0-test:** Similarly,<sup>12</sup>

$$\mathbf{1} - N_a = aa^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|.$$

(Feynman writes this  $\mathbf{1} - N_a$  because he writes  $\mathbf{1} = I$ .)

This has the effect of returning  $|0\rangle$  for input  $|0\rangle$ , but  $\mathbf{0}$  for  $|1\rangle$ .

This is test for  $|0\rangle$ .

(This is the rest of the identity operation.)

¶11. **Universality:** The two operations  $a$  and  $a^*$  are sufficient for creating all  $2 \times 2$  matrices, and therefore all transformations on a single qubit.

Note that

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = waa^* + xa + ya^* + za^*a.$$

<sup>11</sup>This matrix is not the same as that given in F82 and F85, since Feynman uses the basis  $|1\rangle = (1, 0)^T$ ,  $|0\rangle = (0, 1)^T$ .

<sup>12</sup>This matrix is different from that given in F82 and F85, as explained in the previous footnote.

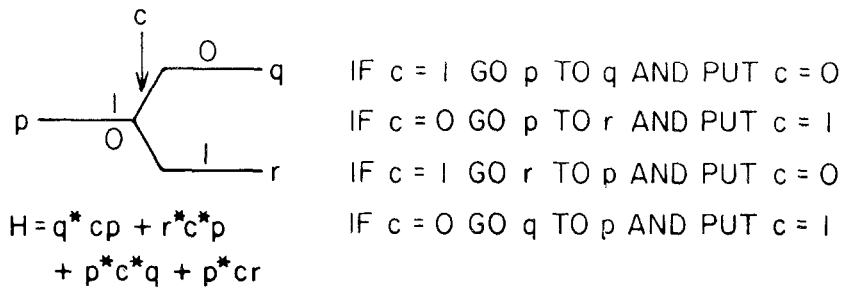


Figure III.40: Switch element. 0/1 annotations on the wires show the  $c$  values. [fig. from F85]

¶12. **Negation:** F writes  $A_a$  for the negation operation applied to  $a$ . Obviously,  $A_a = a + a^*$  (it annihilates  $|1\rangle$  and creates from  $|0\rangle$ ) and  $\mathbf{1} = aa^* + a^*a$  (it passes  $|0\rangle$  and passes  $|1\rangle$ ). Prove that  $A_a A_a = \mathbf{1}$  (exercise).

¶13. **CNOT:** F writes  $A_{a,b}$  for the CNOT operation applied to lines  $a$  and  $b$ .  
 $A_{a,b} = a^*a(b + b^*) + aa^*$ .  
 Notice that this is a tensor product on the register  $|a, b\rangle$ :  
 $A_{a,b} = a^*a \otimes (b + b^*) + aa^* \otimes \mathbf{1}$ .  
 You can write this formula  $N_a \otimes A_b + (\mathbf{1} - N_a) \otimes \mathbf{1}$ . That is, if  $N_a$  detects  $|1\rangle$ , then it negates  $b$ .  
 If  $\mathbf{1} - N_a$  detects  $|0\rangle$ , then it leaves  $b$  alone.

¶14. **CCNOT:** F writes  $A_{ab,c}$  for the CCNOT operation applied to lines  $a$ ,  $b$ , and  $c$ .  
 $A_{ab,c} = \mathbf{1} + a^*ab^*b(c + c^* - \mathbf{1})$  (exercise).  
 This formula is more comprehensible in this form:

$$A_{ab,c} = \mathbf{1} + N_a N_b (A_c - \mathbf{1}).$$

¶15. **SWITCH:** One of Feynman’s universal computers is based on only two logic gates, NOT and SWITCH (Fig. III.40).  
 If  $c = |1\rangle$ , then the “cursor” (locus of control) at  $p$  moves to  $q$ , but if

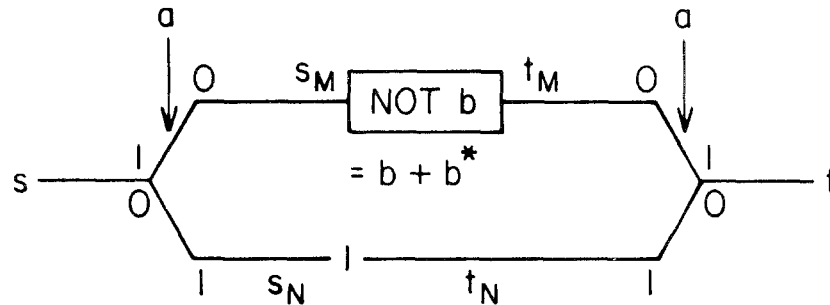


Figure III.41: CNOT implemented by switches. 0/1 annotations on the wires show the  $a$  values. [fig. from F85]

$c = |0\rangle$  it moves to  $r$ .

It also negates  $c$  in the process.

¶16. It's also reversible (see Fig. III.40).

¶17. The switch is a tensor product on  $|c, p, q, r\rangle$ :

$$q^*cp + r^*c^*p + [p^*c^*q + p^*cr].$$

(The bracketed expression is just the complex conjugate of the first part, required for reversibility.) Read the factors in each term from right to left:

(1)  $q^*cp$ : if  $p$  and  $c$  are set, then unset them and set  $q$ .

(2)  $r^*c^*p$ : if  $p$  is set and  $c$  is not set, the unset  $p$  and set  $c$  and  $r$ .

¶18. **CNOT**: Fig. III.41 shows CNOT implemented by switches. This is the controlled-NOT applied to data  $a, b$  and sequenced by cursor atoms  $s, t$  (= start, terminate).

¶19. If  $a = 1$  the cursor state moves along the top line, and if  $a = 0$  along the bottom.

¶20. If it moves along the top, then it applies  $b + b^*$  to negate  $b$  (otherwise leaving it alone).

¶21. In either case, the cursor arrives at the reversed switch, where sets the next cursor atom  $t$ .

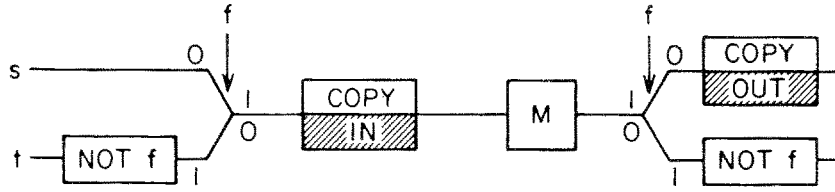


Figure III.42: Garbage clearer. 0/1 annotations on the wires show the  $f$  values. [fig. from F85]

¶22. We can write it

$$H_{a,b}(s, t) = s_M^* a s + t^* a^* t_M + t_M^* (b + b^*) s_M + s_N^* a^* s + t^* a t_N + t_N^* s_N + \text{c.c.},$$

where “c.c.” means to add the complex conjugates of the preceding terms. Read the factors in each term from right to left:

- (1)  $s_M^* a s$ : if  $s$  and  $a$  are set, then unset them and set  $s_M$ .
  - (4)  $s_N^* a^* s$ : if  $s$  is set and  $a$  is unset, then unset  $s$  and set  $s_N$  and  $a$ .
  - (6)  $t_N^* s_N$ : if  $s_N$  is set, then unset it and set  $t_N$ .
  - (3)  $t_M^* (b + b^*) s_M$ : if  $s_M$  is set, then unset it, negate  $b$  and set  $t_M$ .
  - (5)  $t^* a t_N$ : if  $t_N$  and  $a$  are set (as  $a$  must be to get here), then unset them and set  $t$ .
  - (2)  $t^* a^* t_M$ : if  $t_M$  is set and  $a$  is unset (as it must be to get here), then reverse their states and set  $t$ .
- (The  $t_N^* s_N$  term can be eliminated by setting  $t_N = s_N$ .)

### F.1.c GARBAGE CLEARER

- ¶1. Instead of having a separate copy of the machine to clear out the garbage, it’s possible to run the same machine backwards (Fig. III.42).
- ¶2. **Initial state:** An external register IN contains the input, and the output register OUT and all machine registers are all 0s.  
 $s$  is the starting program atom.  
 The flag  $f$  is initially 0.
- ¶3. The  $f = 0$  routes control through the reversed switch (setting  $f = 1$ ) to COPY.

- ¶4. The COPY box uses CNOTs to copy the external input into  $M$
- ¶5.  $M$  operates, generating the result in an internal register.  
 $M$  contains garbage.
- ¶6. The  $f = 1$  flag directs control into the upper branch (resetting  $f = 0$ ), which uses CNOTs to copy the result into the external output register OUT.
- ¶7. Control passes out from the upper branch of the switch down and back into the lower branch, which negates  $f$ , setting  $f = 1$ .
- ¶8. Control passes back into the machine through the lower switch branch (resetting  $f = 0$ ), and backwards through  $M$ , clearing out all the garbage, restoring all the registers to 0s.
- ¶9. It passes backwards through the COPY box, copying the input back from  $M$  to the external input register IN.  
This restores the internal register to 0s.
- ¶10. Control passes out through the lower branch of the left switch (setting  $f = 1$ ), but it negates  $f$  again, so  $f = 0$ .  
It arrives at the terminal program atom  $t$ .
- ¶11. At the end of the process, everything is reset to the initial conditions, except that we have the result in the OUT register.
- ¶12. **Subroutines etc.:** F discusses how to do subroutines and other programming constructs.

## F.2 Benioff's quantum Turing machine

- ¶1. In 1980 Paul Benioff published the first design for a universal quantum computer, which was based on the Turing machine.
- ¶2. **Tape:** The tape is represented by a finite lattice of quantum spin systems with eigenstates corresponding to the tape symbols.  
(Therefore, he cannot implement an open-ended TM tape, but neither can an ordinary digital computer.)
- ¶3. **Head:** The head is a spinless system that moves along the lattice.



- ¶4. **State:** The state of the TM was represented by another spin system.
- ¶5. He defined unitary operators for doing the various operations (e.g., changing the tape).
- ¶6. In 1982 he extended his model to erase the tape, as in Bennett’s model.
- ¶7. **Computation step:** Each step was performed by measuring the tape state under the head and the internal state (thus collapsing them) and using this to control the unitary operator applied to the tape and state.
- ¶8. As a consequence, the computer does not make much use of superposition.

### F.3 Deutsch’s universal quantum computer

This section is based on Deutsch, D., “Quantum theory, the Church-Turing principle, and the universal quantum computer. *Proc. Royal Soc. London A*, **400** (1985), pp. 97–119.

- ¶1. Benioff’s computer is effectively classical; it can be simulated by a classical TM.
- ¶2. Feynman’s construction is not a true universal computer, since you need to construct it for each computation, and it’s not obvious how to get the required dynamical behavior.
- ¶3. Deutsch seeks a broader definition of quantum computation, and a universal quantum computer  $\mathcal{Q}$ .
- ¶4. **Processor:** “The processor consists of  $M$  2-state observables,  $\{\check{n}_i\}$ ” ( $i \in \mathbf{M}$ ), where  $\mathbf{M} = \{0, \dots, M - 1\}$ . Collectively they are called  $\check{\mathbf{n}}$ .
- ¶5. **Memory:** “The memory consists of an infinite sequence  $\{\check{m}_i\}$  ( $i \in \mathbb{Z}$ ) of 2-state observables.” Collectively the sequence is called  $\check{\mathbf{m}}$ .
- ¶6. **Tape position:** An observable  $\check{x}$ , with spectrum  $\mathbb{Z}$ , represents the tape position (address) of the head.

- ¶7. **Computational basis states:** The *computational basis states* have the form:

$$|x; \mathbf{n}; \mathbf{m}\rangle \stackrel{\text{def}}{=} |x; n_0, n_1, \dots, n_{M-1}; \dots, m_{-1}, m_0, m_1, \dots\rangle.$$

Here the eigenvectors are labeled by their eigenvalues  $x$ ,  $\mathbf{n}$ , and  $\mathbf{m}$ .

- ¶8. **Dynamics:** The dynamics of computation is described by a unitary operator  $U$ :

$$|\psi(nT)\rangle = U^n |\psi(0)\rangle.$$

- ¶9. **Initial tape:** Initially, only a finite number of these are prepared in a non-zero state.

$$|\psi(0)\rangle = \sum_m \lambda_m |0; \mathbf{0}; \mathbf{m}\rangle, \text{ where } \sum_m |\lambda_m|^2 = 1,$$

“where only a finite number of the  $\lambda_m$  are non-zero and  $\lambda_m$  vanishes whenever an infinite number of the  $\mathbf{m}$  are non-zero.”

Note that this may be a superposition of initial tapes.

- ¶10. The non-zero entries are the program and its input.
- ¶11. **Unitary operator:** The matrix elements of  $U$  (relating the new state to the current state) have the form:

$$\begin{aligned} \langle x'; \mathbf{n}'; \mathbf{m}' | U | x; \mathbf{n}; \mathbf{m} \rangle \\ = [\delta_{x'}^{x+1} U^+(\mathbf{n}', m'_x | \mathbf{n}, m_x) + \delta_{x'}^{x-1} U^-(\mathbf{n}', m'_x | \mathbf{n}, m_x)] \prod_{y \neq x} \delta_{m'_y}^{m_y}. \end{aligned}$$

$U^+$  and  $U^-$  represent moves to the right and left, respectively.

The first two  $\delta$ s ensure that the tape position cannot move by more than one position in a step.

The final product of deltas ensures that all the other tape positions are unchanged; it's equivalent to:  $\forall y \neq x : m_y = m'_y$ .

- ¶12. The  $U^+$  and  $U^-$  functions define the actual transitions of the machine in terms of the processor state and the symbol under the tape head. Each choice defines a quantum computer  $\mathcal{Q}[U^+, U^-]$ .

- ¶13. **Halting:** The machine cannot be observed before it has halted, since it will generally alter its state.  
Therefore one of the processor's bits is chosen as a halt indicator.  
It can be observed from time to time without affecting the computation.
  
- ¶14. **Power:**  $\mathcal{Q}$  can simulate TMs, but also any other quantum computer to arbitrary precision.  
It can simulate any finitely realizable physical system to arbitrary precision.  
It can simulate some physical systems that go beyond the power of TMs (hypercomputation).