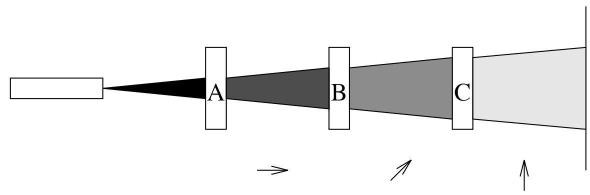

Figure III.4: Fig. from Rieffel & Polak (2000).

B.4 Superposition

A simple experiment demonstrates quantum effects that can not be explained by classical physics (see Fig. III.4). Suppose we have three polarizing filters, A, B, and C, polarized horizontally, 45°, and vertically, respectively. Place the horizontal filter A between a strong light source, such as a laser, and a screen. The light intensity is reduced by one half and the light is horizontally polarized. (Note: Since the light source is unpolarized, i.e., it has all polarizations, the resulting intensity would be much less than one half if the filter allowed only exactly horizontally polarized light through, as would be implied by a sieve model of polarization.) Next insert filter C, polarized vertically, and the intensity drops to zero. This is not surprising, since the filters are cross-polarized. Finally, insert filter B, polarized diagonally, between A and C, and surprisingly some light (about 1/8 intensity) will return! This can't be explained by the sieve model. How can putting in more filters increase the light intensity?

Quantum mechanics provides a simple explanation of the this effect; in fact, it's exactly what we should expect. A photon's polarization state can be represented by a unit vector pointing in appropriate direction. Therefore, arbitrary polarization can be expressed by $a|0\rangle + b|1\rangle$ for any two basis vectors $|0\rangle$ and $|1\rangle$, where $|a|^2 + |b|^2 = 1$.

A polarizing filter measures a state with respect to a basis that includes a vector parallel to its polarization and one orthogonal to it. The effect of filter A is the projector $P_{\rm A} = |{\rightarrow}\rangle\langle{\rightarrow}|$. To get the probability amplitude, apply it to $|\psi\rangle \stackrel{\rm def}{=} a|{\rightarrow}\rangle + b|\uparrow\rangle$:

$$p(\mathrm{A}) = |\langle \rightarrow | \psi \rangle|^2 = |\langle \rightarrow |(a|{\rightarrow}\rangle + b|{\uparrow}\rangle)|^2 = |a\langle{\rightarrow}|{\rightarrow}\rangle + b\langle{\rightarrow}|{\uparrow}\rangle|^2 = |a|^2.$$

So with probability $|a|^2$ we get $|{\rightarrow}\rangle$ (recall Eqn. III.1, p. 84). So if the polarizations are randomly distributed from the source, half will get through

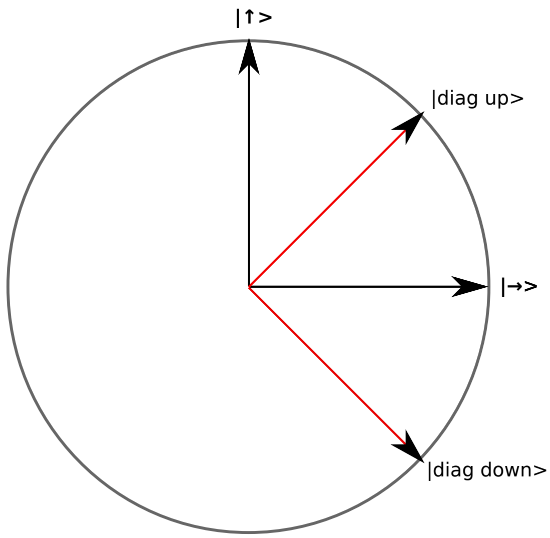

Figure III.5: Alternative polarization bases for measuring photons (black = rectilinear basis, red = diagonal basis). Note $|\nearrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\rightarrow\rangle)$ and $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\nearrow\rangle + |\searrow\rangle)$.

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and all of them will be in state $|\rightarrow\rangle$. Why one half? Note that $a = \cos\theta$, where θ is the angle between $|\psi\rangle$ and $|\rightarrow\rangle$, and that

$$\langle a^2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cos^2\theta \, \mathrm{d}\theta = \frac{1}{2}.$$

When we insert filter C we are measuring with the projector $P_{\mathrm{C}} = |\uparrow\rangle\langle\uparrow|$ and the result is 0, as expected:

$$p(\mathrm{AC}) = |\langle\uparrow|\rightarrow\rangle|^2 = 0.$$

Now insert the diagonal filter B between the horizontal and vertical filters A and C. Filter B measures with respect to the projector $\{|\nearrow\rangle, |\searrow\rangle\}$ basis (see Fig. III.5). Transmitted light is given by the projector $P_{\mathrm{B}} = |\nearrow\rangle\langle\nearrow|$. To find the result of applying filter B to the horizontally polarized light emerging from filter A, we must express $|\rightarrow\rangle$ in the diagonal basis:

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\nearrow\rangle + |\searrow\rangle).$$

So if filter B is $|\nearrow\rangle\langle\nearrow|$ we get $|\nearrow\rangle$ photons passing through filter B with probability 1/2:

$$p(\mathrm{B}) = |\langle\nearrow|\rightarrow\rangle|^2 = \left|\langle\nearrow|\left[\frac{1}{\sqrt{2}}(|\nearrow\rangle + |\searrow\rangle\right]\right|^2 = \left|\frac{1}{\sqrt{2}}\langle\nearrow|\nearrow\rangle = \frac{1}{\sqrt{2}}\langle\nearrow|\searrow\rangle\right|^2 = \frac{1}{2}.$$

Hence, the probability of source photons passing though filters A and B is $p(\mathrm{AB}) = p(\mathrm{A})p(\mathrm{B}) = 1/4$.

The effect of filter C, then, is to measure $|\nearrow\rangle$ by projecting against $|\uparrow\rangle$. Note that

$$|\nearrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\rightarrow\rangle).$$

The probability of these photons getting through filter C is

$$|\langle\rightarrow|\nearrow\rangle|^2 = \left|\langle\rightarrow|\left[\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\rightarrow\rangle)\right]\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

Therefore we get $|\rightarrow\rangle$ with another 1/2 decrease in intensity (so 1/8 overall).

B.5 No-cloning theorem

Copying and erasing are two of the fundamental (blackboard-inspired) operations of conventional computing. However, the *No-cloning Theorem* of quantum mechanics states that it is impossible to copy the state of a qubit. To see this, assume on the contrary that we have a unitary transformation U that does the copying, so that $U(|\psi\rangle \otimes |c\rangle) = |\psi\rangle \otimes |\psi\rangle$, where $|c\rangle$ is an arbitrary constant qubit (actually, $|c\rangle$ can be any quantum state). That is, $U|\psi c\rangle = |\psi\psi\rangle$. Next suppose that $|\psi\rangle = a|0\rangle + b|1\rangle$. By the linearity of U:

$$\begin{aligned}
U\ |\psi\rangle|c\rangle &= U\ (a|0\rangle + b|1\rangle)|c\rangle \\
&= U(a|0\rangle|c\rangle + b|1\rangle|c\rangle) && \text{distrib. of tensor prod.} \\
&= U(a|0c\rangle + b|1c\rangle) \\
&= a(U|0c\rangle) + b(U|1c\rangle) && \text{linearity} \\
&= a|00\rangle + b|11\rangle && \text{copying property.}
\end{aligned}$$

On the other hand, by expanding $|\psi\psi\rangle$ we have:

$$\begin{aligned}
U\ |\psi c\rangle &= |\psi\psi\rangle \\
&= (a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle) \\
&= a^2|00\rangle + ba|10\rangle + ab|01\rangle + b^2|11\rangle.
\end{aligned}$$

Note that these two expansions cannot be made equal in general, so no such unitary transformation exists. Cloning is possible only in the special cases $a = 0, b = 1$ or $a = 1, b = 0$, that is, only where we know that we are cloning a determinate (classical) basis state. The inability to simply copy a quantum state is one of the characteristics of quantum computation that makes it significantly different from classical computation.

B.6 Entanglement

B.6.a Entangled and decomposable states

The possibility of *entangled quantum states* is one of the most remarkable characteristics distinguishing quantum from classical systems. Suppose that $\mathcal{H}'$ and $\mathcal{H}''$ are the state spaces of two quantum systems. Then $\mathcal{H} = \mathcal{H}' \otimes \mathcal{H}''$ is the state space of the *composite system* (Postulate 4). For simplicity, suppose that both spaces have the basis $\{|0\rangle, |1\rangle\}$. Then $\mathcal{H}' \otimes \mathcal{H}''$ has the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. (Recall that $|01\rangle = |0\rangle \otimes |1\rangle$, etc.) Arbitrary elements of $\mathcal{H}' \otimes \mathcal{H}''$ can be written in the form

$$\sum_{j,k=0,1} c_{jk}|jk\rangle = \sum_{j,k=0,1} c_{jk}\ |j'\rangle \otimes |k''\rangle.$$

Sometimes the state of the composite systems can be written as the tensor product of the states of the subsystems, $|\psi\rangle = |\psi'\rangle \otimes |\psi''\rangle$. Such a state is called a *separable*, *decomposable* or *product state*. In other cases the state cannot be decomposed, in which case it is called an *entangled state*

For an example of an entangled state, consider the *Bell state* $|\beta_{01}\rangle$, which might arise from a process that produced two particles with opposite spin (but without determining which is which):

$$|\beta_{01}\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \stackrel{\text{def}}{=} |\Phi^+\rangle. \tag{III.2}$$

(The notations $|\beta_{01}\rangle$ and $|\Phi^+\rangle$ are both used.) Note that the states $|01\rangle$ and $|10\rangle$ both have probability 1/2. Such a state might arise, for example, from a process that emits two particles with opposite spin angular momentum in order to preserve conservation of spin angular momentum.

To show that $|\beta_{01}\rangle$ is entangled, we need to show that it cannot be decomposed, that is, that we cannot write $|\beta_{01}\rangle = |\psi'\rangle \otimes |\psi''\rangle$, for two state vectors $|\psi'\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\psi''\rangle = b_0|0\rangle + b_1|1\rangle$. Let's try a separation or decomposition:

$$|\beta_{01}\rangle \stackrel{?}{=} (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle).$$

Multiplying out the RHS yields:

$$a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle.$$

Therefore we must have $a_0b_0 = 0$ and $a_1b_1 = 0$. But this implies that either $a_0b_1 = 0$ or $a_1b_0 = 0$ (as opposed to $1/\sqrt{2}$), so the decomposition is impossible.

For an example of a decomposable state, consider $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Writing out the product $(a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$ as before, we require $a_0b_0 = a_0b_1 = a_1b_0 = a_1b_1 = \frac{1}{2}$. This is satisfied by $a_0 = a_1 = b_0 = b_1 = \frac{1}{\sqrt{2}}$, therefore the state is decomposable.

In addition to Eq. III.2, the other three Bell states are defined:

$$|\beta_{00}\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \stackrel{\text{def}}{=} |\Psi^+\rangle, \tag{III.3}$$

$$|\beta_{10}\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \stackrel{\text{def}}{=} |\Psi^-\rangle, \tag{III.4}$$

$$|\beta_{11}\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \stackrel{\text{def}}{=} |\Phi^-\rangle. \tag{III.5}$$

The Ψ states have two identical qubits, the Φ states have opposite qubits. The $+$ superscript indicates they are added, the $-$ that they are subtracted. The general definition is:

$$|\beta_{xy}\rangle = \frac{1}{\sqrt{2}}(|0, y\rangle + (-1)^x |1, \neg y\rangle).$$

Remember this useful formula! The Bell states are orthogonal and in fact constitute a basis for $\mathcal{H}' \otimes \mathcal{H}''$ (exercise).

B.6.b EPR paradox

The EPR Paradox was proposed by Einstein, Podolsky, and Rosen in 1935 to show problems in quantum mechanics. Our discussion here will be informal.

Suppose a source produces an entangled *EPR pair* (or *Bell state*) $|\Psi^+\rangle = |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and the entangled particles are sent to Alice and Bob. If Alice measures her particle and gets $|0\rangle$, then that collapses the state to $|00\rangle$, and so Bob will have to get $|0\rangle$ if he measures his particle. Likewise, if Alice happens to get $|1\rangle$, Bob is also required to get $|1\rangle$ if he measures. This happens instantaneously (but it does not permit faster-than-light communication, as explained below).

One explanation is that there is some internal state in the particles that will determine the result of the measurement. Both particles have the same internal state. Such *hidden-variable theories* of quantum mechanics assume that particles are "really" in some definite state and that superposition reflects our ignorance of its state. However, they cannot explain the results of measurements in different bases. In 1964 John Bell showed that any local hidden variable theory would lead to measurements satisfying a certain inequality (Bell's inequality). Actual experiments, which have been conducted over tens of kilometers, violate Bell's inequality. Thus local hidden variable theories cannot be correct.

Another explanation is that Alice's measurement affects Bob's (or vice versa, if Bob measures first). These are called *causal theories*. According to relativity theory, however, in some frames of reference Alice's measurement comes first, and in other frames, Bob's comes first. Therefore there is no consistent cause-effect relation. This is why Alice and Bob cannot use entangled pairs to communicate.