H Exercises

Exercise III.1 Compute the probability of measuring $|0\rangle$ and $|1\rangle$ for each of the following quantum states:

1. $0.6|0\rangle + 0.8|1\rangle$.
2. $\frac{1}{\sqrt{3}}|0\rangle + \sqrt{2/3}|1\rangle$.
3. $\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$.
4. $-\frac{1}{25}(24|0\rangle - 7|1\rangle)$.
5. $-\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/6}}{\sqrt{2}}|1\rangle$.

Exercise III.2 Suppose that a two-qubit register is in the state

$$|\psi\rangle = \frac{3}{5}|00\rangle - \frac{\sqrt{7}}{5}|01\rangle + \frac{e^{i\pi/2}}{\sqrt{5}}|10\rangle - \frac{2}{5}|11\rangle.$$

1. Suppose we measure just the first qubit. Compute the probability of measuring a $|0\rangle$ or a $|1\rangle$ and the resulting register state in each case.
2. Do the same, but supposing instead that we measure just the second qubit.

Exercise III.3 Prove that projectors are idempotent, that is, $P^2 = P$.

Exercise III.4 Prove that a normal matrix is Hermitian if and only if it has real eigenvalues.

Exercise III.5 Prove that $U(t) \stackrel{\text{def}}{=} \exp(-iHt/\hbar)$ is unitary.

Exercise III.6 Use spectral decomposition to show that $K = -i\log(U)$ is Hermitian for any unitary U, and thus $U = \exp(iK)$ for some Hermitian K.

Exercise III.7 Show that the commutators ($[L, M]$ and $\{L, M\}$) are bilinear (linear in both of their arguments).

Exercise III.8 Show that $[L, M]$ is anticommutative, i.e., $[M, L] = -[L, M]$, and that $\{L, M\}$ is commutative.

Exercise III.9 Show that $LM = \frac{[L,M]+\{L,M\}}{2}$.

Exercise III.10 Show that the four Bell states are orthonormal (i.e., both orthogonal and normalized).

Exercise III.11 Prove that $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is entangled.

Exercise III.12 What is the effect of Y (imaginary definition) on the computational basis vectors? What is its effect if you use the real definition (C.2.a, p. 105)?

Exercise III.13 Prove that I, X, Y, and Z are unitary. Use either the imaginary or real definition of Y (C.2.a, p. 105).

Exercise III.14 What is the matrix for H in the *sign basis*?

Exercise III.15 Show that the X, Y, Z and H gates are Hermitian (their own inverses) and prove your answers. Use either the imaginary or real definition of Y (C.2.a, p. 105).

Exercise III.16 Prove the following useful identities:

$$HXH = Z, HYH = -Y, HZH = X.$$

Exercise III.17 Show (using the real definition of Y, C.2.a, p. 105):
$|0\rangle\langle 0| = \frac{1}{2}(I+Z), |0\rangle\langle 1| = \frac{1}{2}(X-Y), |1\rangle\langle 0| = \frac{1}{2}(X+Y), |1\rangle\langle 1| = \frac{1}{2}(I-Z)$.

Exercise III.18 Prove that the Pauli matrices span the space of 2×2 matrices.

Exercise III.19 Prove $|\beta_{xy}\rangle = (P \otimes I)|\beta_{00}\rangle$, where $xy = 00, 01, 11, 10$ for $P = I, X, Y, Z$, respectively.

Exercise III.20 Suppose that P is one of the Pauli operators, but you don't know which one. However, you are able to pick a 2-qubit state $|\psi_0\rangle$ and operate on it, $|\psi_1\rangle = (P \otimes I)|\psi_0\rangle$. Further, you are able to select a unitary operation U to apply to $|\psi_1\rangle$, and to measure the 2-qubit result, $|\psi_2\rangle = U|\psi_1\rangle$, in the computational basis. Select $|\psi_0\rangle$ and U so that you can determine with certainty the unknown Pauli operator P.

Exercise III.21 What is the matrix for CNOT in the standard basis? Prove your answer.

Exercise III.22 Show that CNOT does not violate the No-cloning Theorem by showing that, in general, $\text{CNOT}|\psi\rangle|0\rangle \neq |\psi\rangle|\psi\rangle$. Under what conditions does the equality hold?

Exercise III.23 What quantum state results from

$$\text{CNOT}(H \otimes I)\,\frac{1}{2}(c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle)?$$

Exercise III.24 What is the matrix for CCNOT in the standard basis? Prove your answer.

Exercise III.25 Use a single Toffoli gate to implement each of NOT, NAND, and XOR.

Exercise III.26 Use Toffoli gates to implement FAN-OUT. FAN-OUT would seem to violate the No-cloning Theorem, but it doesn't. Explain why.

Exercise III.27 Design a quantum circuit to transform $|000\rangle$ into the entangled state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

Exercise III.28 Show that $|+\rangle, |-\rangle$ is an ON basis.

Exercise III.29 Prove:

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \\ |1\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \end{aligned}$$

Exercise III.30 What are the possible outcomes (probabilities and resulting states) of measuring $a|+\rangle + b|-\rangle$ in the *computational basis* (of course, $|a|^2 + |b|^2 = 1$)?

Exercise III.31 Prove that $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$.

Exercise III.32 Prove:

$$\begin{aligned} H(a|0\rangle + b|1\rangle) &= a|+\rangle + b|-\rangle, \\ H(a|+\rangle + b|-\rangle) &= a|0\rangle + b|1\rangle. \end{aligned}$$

Exercise III.33 Prove $H = (X + Z)/\sqrt{2}$.

Exercise III.34 Prove Eq. III.18 (p. 111).

Exercise III.35 Show that three successive CNOTs, connected as in Fig. III.11 (p. 110), will swap two qubits.

Exercise III.36 Recall the conditional selection between two operators (C.3, p. 111): $|0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1$. Suppose the control bit is a superposition $|\chi\rangle = a|0\rangle + b|1\rangle$. Show that:

$$(|0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1)|\chi, \psi\rangle = a|0, U_0\psi\rangle + b|1, U_1\psi\rangle.$$

Exercise III.37 Show that the 1-bit full adder (Fig. III.15, p. 113) is correct.

Exercise III.38 Show that the operator U_f is unitary:

$$U_f|x, y\rangle \stackrel{\text{def}}{=} |x, y \oplus f(x)\rangle,$$

Exercise III.39 Verify the remaining superdense encoding transformations in Sec. C.6.a (p. 117).

Exercise III.40 Verify the remaining decoding cases for quantum teleportation Sec. C.6.b (p. 121).

Exercise III.41 Confirm the quantum teleportation circuit in Fig. III.21 (p. 122).

Exercise III.42 Complete the following step from the derivation of the Deutsch-Jozsa algorithm (Sec. D.1, p. 129):

$$H|x\rangle = \sum_{z \in \mathbf{2}} \frac{1}{\sqrt{2}} (-1)^{xz} |z\rangle.$$

Exercise III.43 Show that $\text{CNOT}(H \otimes I) = (I \otimes H)C_Z H^{\otimes 2}$, where C_Z is the controlled-Z gate.

Exercise III.44 Show that the Fourier transform matrix (Eq. III.25, p. 137, Sec. D.3.a) is unitary.

Exercise III.45 Prove the claim on page 152 (Sec. D.4.b) that D is unitary.

Exercise III.46 Prove the claim on page 152 (Sec. D.4.b) that

$$WR'W = \begin{pmatrix} \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \end{pmatrix}.$$

Exercise III.47 Show that if there are s solutions x such that $P(x) = 1$, then $\frac{\pi\sqrt{N/s}}{4}$ is the optimal number of iterations in Grover's algorithm.

Exercise III.48 Design a quantum gate array for the following syndrome extraction operator (Sec. D.5.d, p. 162):

$$S|x_3, x_2, x_1, 0, 0, 0\rangle \stackrel{\text{def}}{=} |x_3, x_2, x_1, x_1 \oplus x_2, x_1 \oplus x_3, x_2 \oplus x_3\rangle.$$

Exercise III.49 Design a quantum gate array to apply the appropriate error correction for the extracted syndrome as given in Sec. D.5.d, p. 162:

bit flipped	syndrome	error correction
none	$\|000\rangle$	$I \otimes I \otimes I$
1	$\|110\rangle$	$I \otimes I \otimes X$
2	$\|101\rangle$	$I \otimes X \otimes I$
3	$\|011\rangle$	$X \otimes I \otimes I$

Exercise III.50 Design encoding, syndrome extraction, and error correction quantum circuits for the code $|0\rangle \mapsto |+++\rangle$, $|1\rangle \mapsto |---\rangle$ to correct single phase flip (Z) errors.

Exercise III.51 Prove that $A_a A_a = \mathbf{1}$ (Sec. F.1.b).

Exercise III.52 Prove that $A_{ab,c} = \mathbf{1} + a^\dagger a b^\dagger b(c + c^\dagger - \mathbf{1}) = \mathbf{1} + N_a N_b (A_c - \mathbf{1})$ is a correct definition of CCNOT by showing how it transforms the quantum register $|a, b, c\rangle$ (Sec. F.1.b).

Exercise III.53 Show that the following definition of Feynman's switch is unitary (Sec. F.1.b):

$$q^\dagger c p + r^\dagger c^\dagger p + p^\dagger c^\dagger q + p^\dagger c r.$$