Solution of *k*-SAT by Analog Computation

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COSC 494/594 Unconventional Computation

Sources

- Based on a dynamical system for solving *k*-SAT by M. Ercsey-Ravasz and colleagues:
  - B. Molnár and M. Ercsey-Ravasz, “Asymmetric continuous-time neural networks without local traps for solving constraint satisfaction problems,” *PLoS ONE*, vol. 8, no. 9, p. e73400, 2013.
  - R. Sumi, B. Molnár, and M. Ercsey-Ravasz, “Robust optimization with transiently chaotic dynamical systems,” *EPL* (*Europhysics Letters*), vol. 106, p. 40002, 2014.
- Analog algorithm and circuit implementation in:

  Brasford, D., Smith, J. M., Connor, R. J., MacLennan, B. J., Holleman, J. “The Impact of Analog Computational Error on an Analog Boolean Satisfiability Solver,” *IEEE International Symposium for Circuits and Systems 2016*, Montreal, Canada, May 2016.

Example *k*-SAT Problem

$$(X_1 \lor X_3 \lor X_4) \land (\overline{X_2} \lor X_3 \lor X_4) \land (X_2 \lor X_4 \lor \overline{X_5})$$

- *N* = 5 variables, *M* = 3 clauses, *k* = 3 literals in each
- Constraint density $\alpha = M/N$

Variables

- Solution variables, $s_i \in [-1, 1]$
  - Negative values represent Boolean 0
  - Positive values represent Boolean 1
- Auxiliary variables, $a_m \in [0, 1]$
  - Indicate "urgency" of satisfying a clause
- Constraint matrix, $c_{mi} \in \{-1, 0, 1\}$
  - $c_{mi} = 1$, if X_i positive in clause m
  - $c_{mi} = -1$, if X_i negative in clause m
  - $c_{mi} = 0$, if X_i not in clause m
- Note that we want $\sum_i c_{mi} s_i$ to be positive

Solution Squashing Function

$$f(s) = (|s+1| - |s-1|)/2 = \begin{cases} -1 & \text{if } s < -1, \\ s & \text{if } -1 \leq s \leq 1, \\ +1 & \text{if } s > 1. \end{cases}$$

- Keeps solution variables bounded

Auxiliary Squashing Function

$$g(a) = (1 + |a| - |1 - a|)/2 = \begin{cases} 0 & \text{if } a < 0, \\ a & \text{if } 0 \leq a \leq 1, \\ +1 & \text{if } a > 1. \end{cases}$$

- Keeps auxiliary variables bounded

Dynamics of Solution Variables

$$\dot{s}_i(t) = -s_i(t) + Af[s_i(t)] + \sum_{m=1}^{M} c_{mi} g[a_m(t)]$$

- *A* is self-coupling parameter
- Summation tends to force s_i to solution, weighted by urgency

Dynamics of Auxiliary Variables

$$\dot{a}_m(t) = -a_m(t) + Bg[a_m(t)] - \sum_{i=1}^{N} c_{mi} f[s_i(t)] + 1 - k$$

- *B* is a self-coupling parameter
- a_m decreases to the extent clause *m* is satisfied

Asymptotic Behavior

- Molnár and Ercsey-Ravasz prove: the only stable fixed points of the system are solutions to the problem
- They give numerical evidence that there are no limit cycles
  - provided A and B are in appropriate range
- Hard instances exhibit transient chaotic behavior

Bounds on Variables

$$|s_i(t)| \leq 1 + A + \sum_m |c_{mi}|$$

$$-2k \leq a_m(t) \leq 2 + B$$

- provided they are initially in appropriate ranges:

  $|s_i(0)| \leq 1$, and $0 \leq a_m(0) \leq 1$

- Important for analog implementation

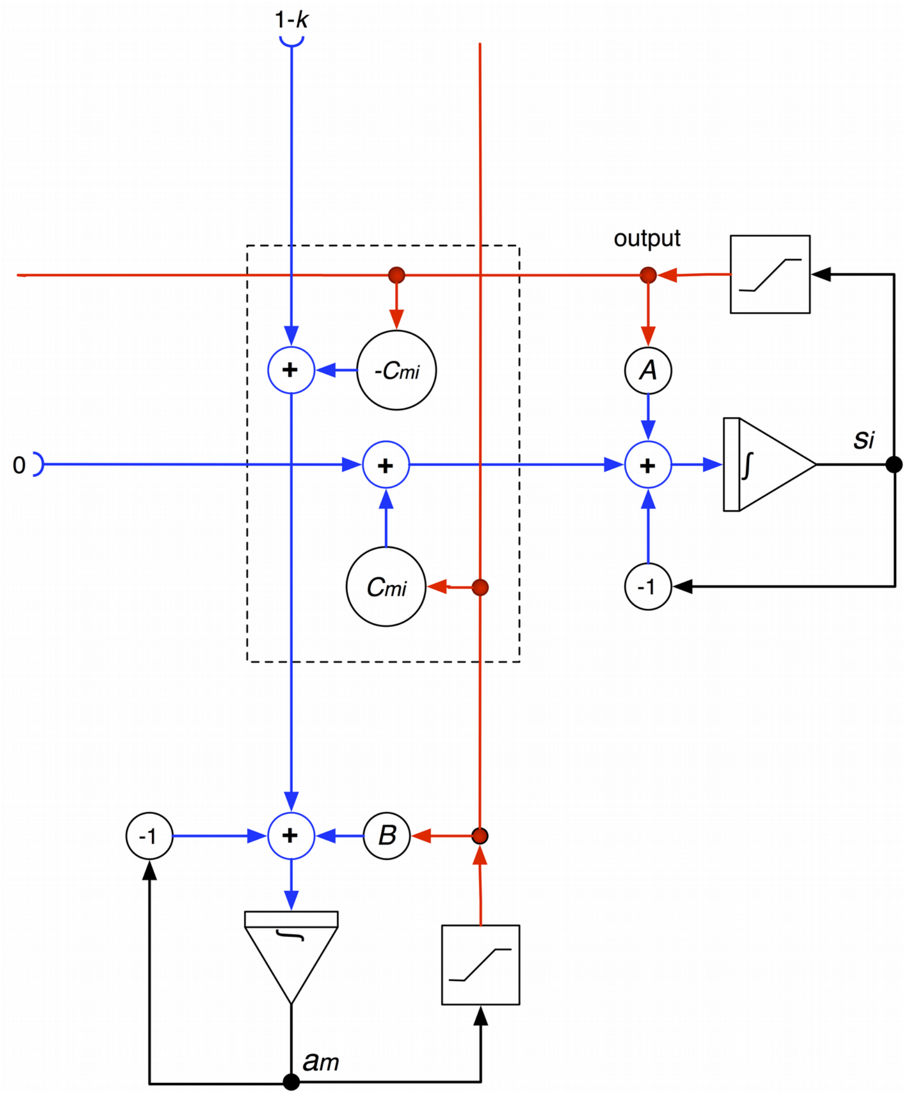
Pseudo-Energy Function

$$E[f(\mathbf{s})] = \mathbf{K}^{\mathrm{T}}\mathbf{K}$$

$$\text{where } K_m = 2^{-k} \prod_{i=1}^{N} [1 - c_{mi} f(s_i)]$$

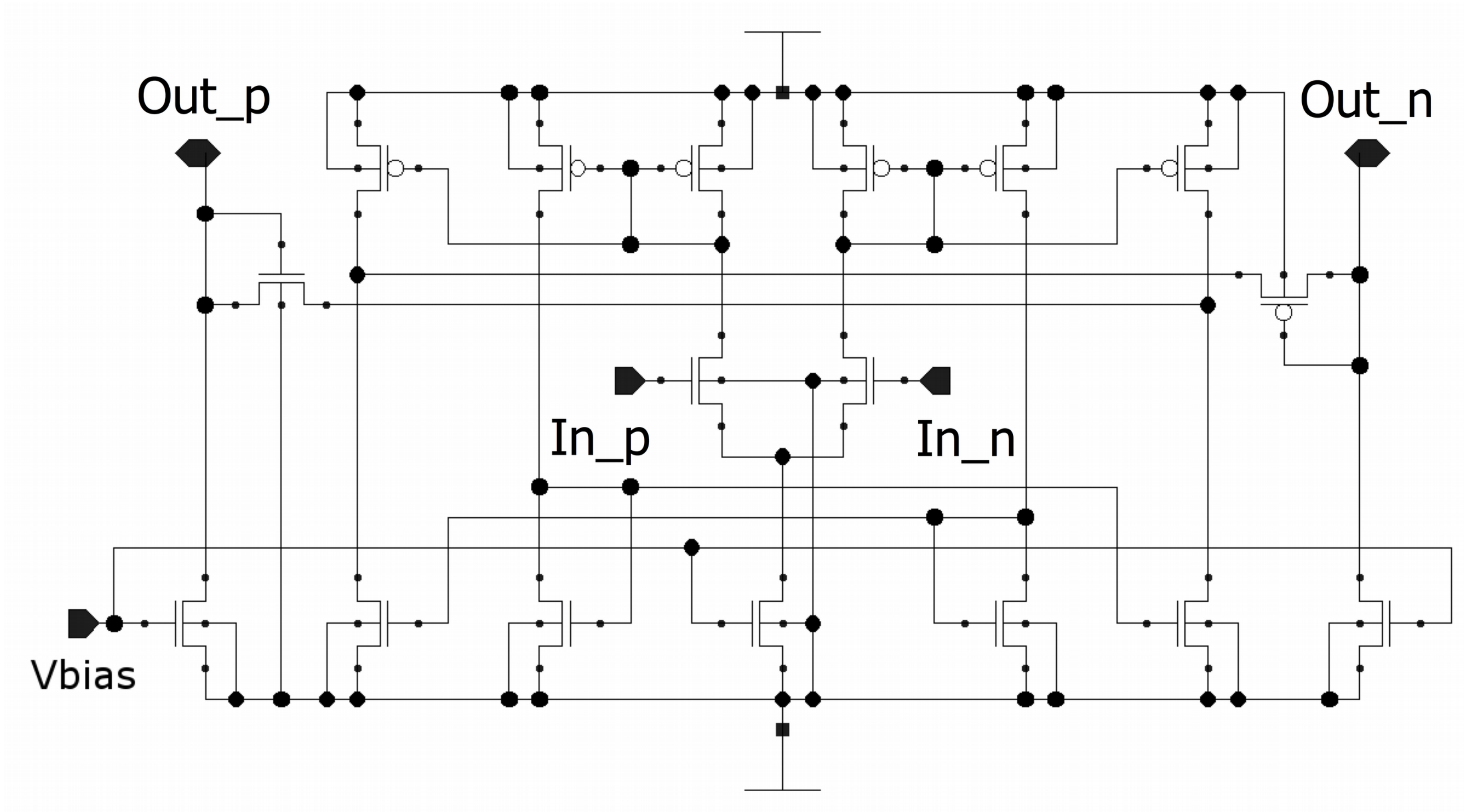
- Increases with number of unsatisfied clauses
- Bracketed expression is 0 in satisfied clauses
- Not a Lyapunov function (does not decrease monotonically)

Analog Algorithm

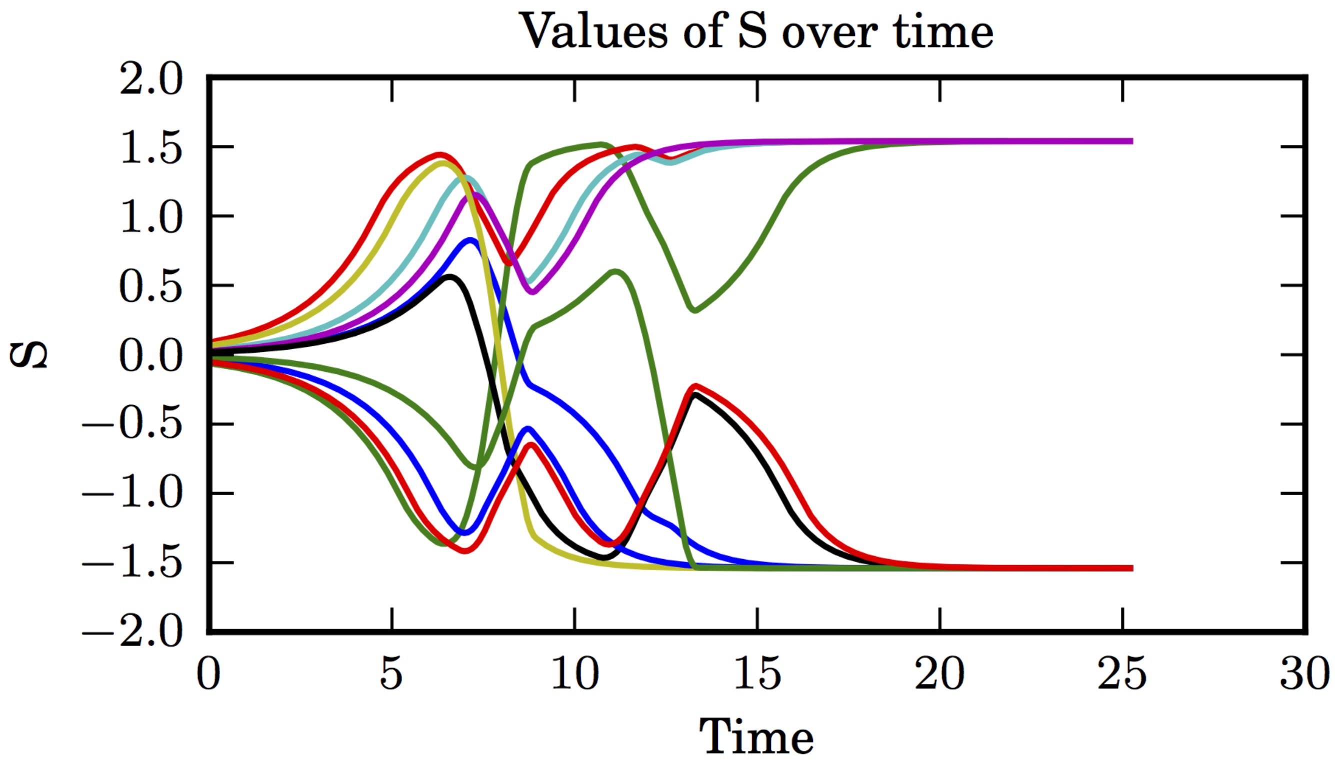


- M integrators for a_m
- N integrators for s_i
- Instance programmed by setting c_{mi} and $-c_{mi}$ connections
- Dotted cell reproduced *MN* times
- Integrators initialized to small values to start computation

Schematic of *g*(*a*) Cell

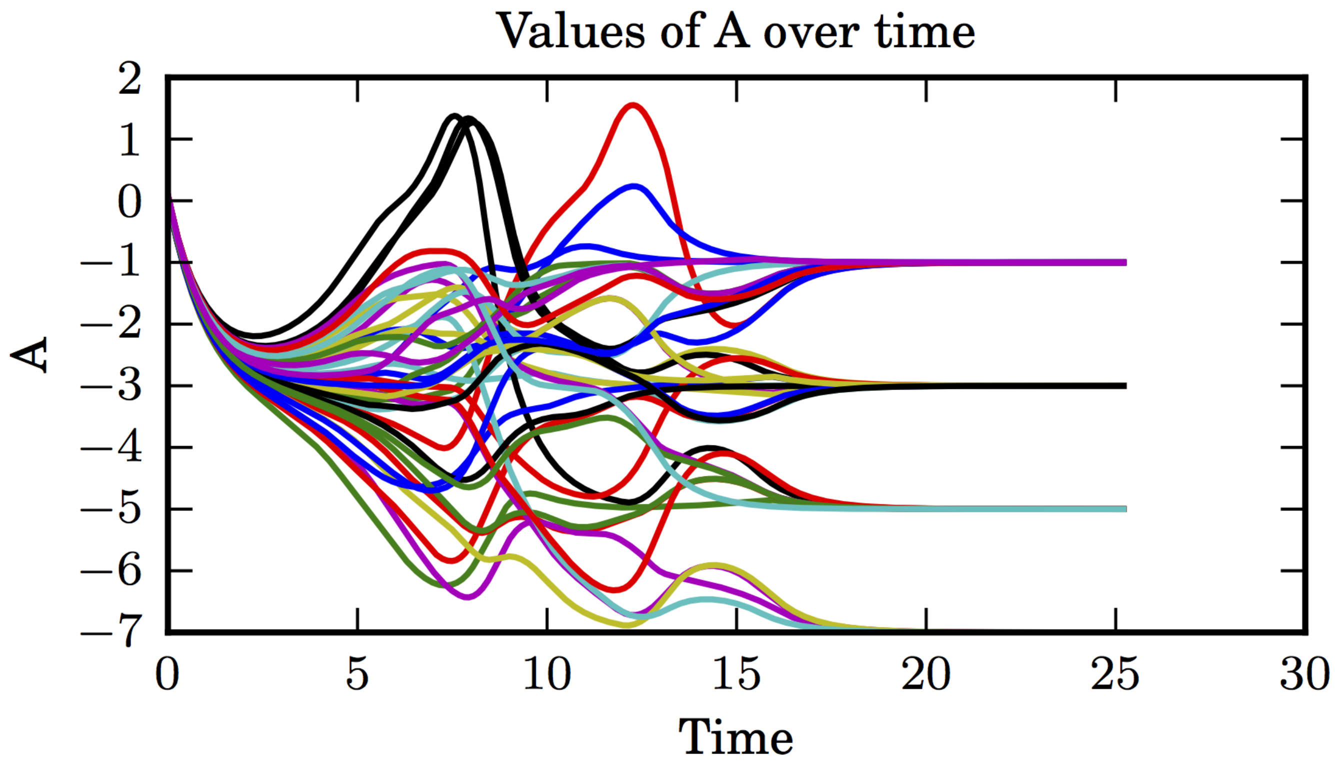


Evolution of Solution Variables



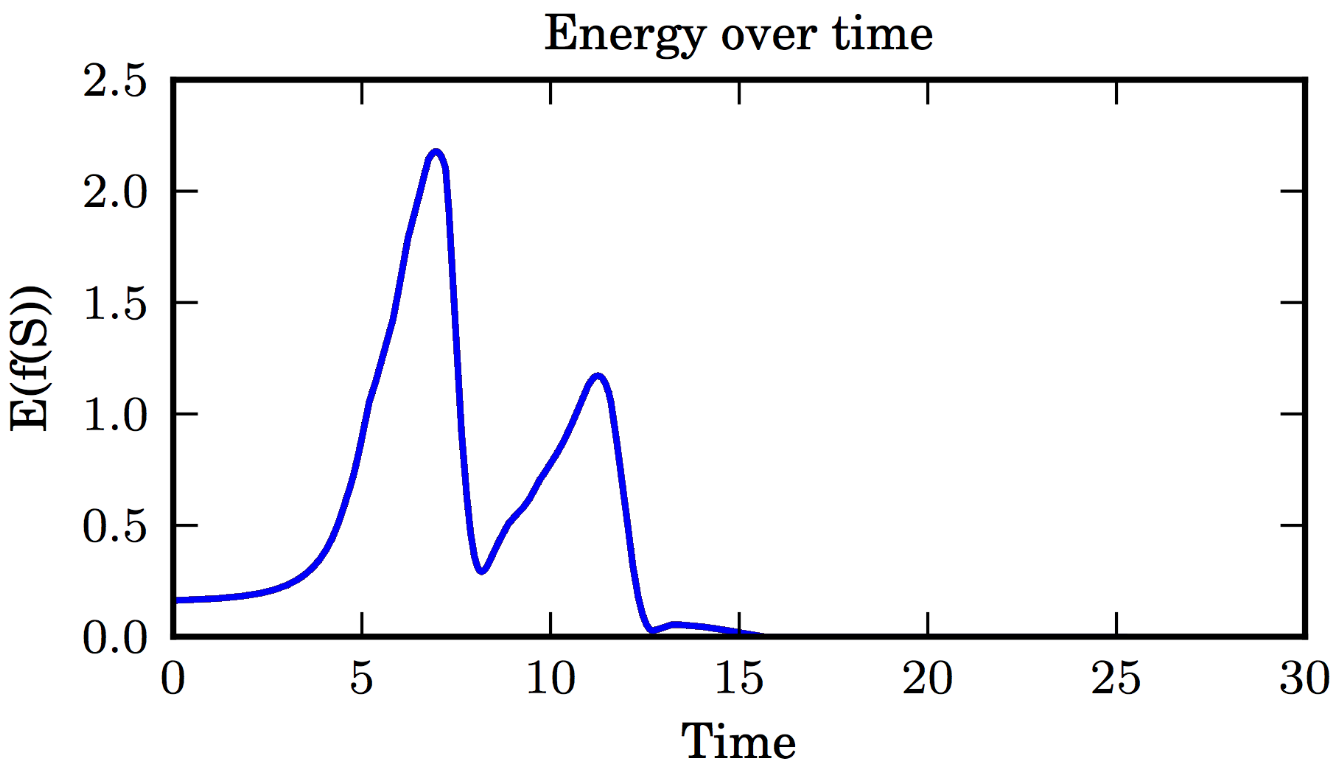
$N = 10$, $k = 4$, $\alpha = 4$

Evolution of Auxiliary Variables



$N = 10$, $k = 4$, $\alpha = 4$

Evolution of Pseudo-Energy



$N = 10$, $k = 4$, $\alpha = 4$

Observations

- This particular algorithm has exponential analog-time perfomance
  - other similar analog algorithms are much more efficient (Ercsey-Ravasz & Toroczkai, 2011)
- Deep theoretical connection between chaotic dynamical systems and hard instances of SAT
  - Turbulence and computational intractability
- One of many examples of analog solutions of discrete problems

References

- M. Ercsey-Ravasz and Z. Toroczkai, “Optimization hardness as transient chaos in an analog approach to constraint satisfaction,” *Nature Physics*, vol. 7, pp. 966–970, 2011.
- B. Molnár and M. Ercsey-Ravasz, “Asymmetric continuous-time neural networks without local traps for solving constraint satisfaction problems,” *PLoS ONE*, vol. 8, no. 9, p. e73400, 2013.
- R. Sumi, B. Molnár, and M. Ercsey-Ravasz, “Robust optimization with transiently chaotic dynamical systems,” *EPL* (*Europhysics Letters*), vol. 106, p. 40002, 2014.
- Brasford, D., Smith, J. M., Connor, R. J., MacLennan, B. J., Holleman, J. “The Impact of Analog Computational Error on an Analog Boolean Satisfiability Solver,” *IEEE International Symposium for Circuits and Systems 2016*, Montreal, Canada, May 2016.
- X. Yin, B. Sedighi, M. Varga, M. Ercsey-Ravasz, Z. Toroczkai, X. Hu, “Efficient Analog Circuits for Boolean Satisfiability,” arXiv:1606.07467v1.