Exercise III.19 Prove the following useful identities:

$$HXH = Z, HYH = -Y, HZH = X.$$

Exercise III.20 Show (using the real definition of Y, C.2.a, p. 112):
$|0\rangle\langle 0| = \frac{1}{2}(I + Z), |0\rangle\langle 1| = \frac{1}{2}(X - Y), |1\rangle\langle 0| = \frac{1}{2}(X + Y), |1\rangle\langle 1| = \frac{1}{2}(I - Z)$.

Exercise III.21 Prove that the Pauli matrices span the space of 2×2 matrices.

Exercise III.22 Prove $|\beta_{xy}\rangle = (P \otimes I)|\beta_{00}\rangle$, where $xy = 00, 01, 11, 10$ for $P = I, X, Y, Z$, respectively.

Exercise III.23 Suppose that P is one of the Pauli operators, but you don't know which one. However, you are able to pick a 2-qubit state $|\psi_0\rangle$ and operate on it, $|\psi_1\rangle = (P \otimes I)|\psi_0\rangle$. Further, you are able to select a unitary operation U to apply to $|\psi_1\rangle$, and to measure the 2-qubit result, $|\psi_2\rangle = U|\psi_1\rangle$, in the computational basis. Select $|\psi_0\rangle$ and U so that you can determine with certainty the unknown Pauli operator P.

Exercise III.24 What is the matrix for CNOT in the standard basis? Prove your answer.

Exercise III.25 Show that CNOT does not violate the No-cloning Theorem by showing that, in general, $\text{CNOT}|\psi\rangle|0\rangle \neq |\psi\rangle|\psi\rangle$. Under what conditions does the equality hold?

Exercise III.26 What quantum state results from

$$\text{CNOT}(H \otimes I)\ \frac{1}{2}(c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle)?$$

Express the result in the computational basis.

Exercise III.27 Compute $(Y \otimes I)\text{CNOT}(H \otimes I)|00\rangle$. Show your work.

Exercise III.28

1. Compute $(H \otimes I \otimes I)(\text{CNOT} \otimes I)[(\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle) \otimes |\beta_{00}\rangle$.
2. Give the probabilities and resulting states for measuring the first two qubits in the computational basis.

3. Apply Z to the state resulting from measuring $|10\rangle$.

Exercise III.29 What is the matrix for CCNOT in the standard basis? Prove your answer.

Exercise III.30 Use a single Toffoli gate to implement each of NOT, NAND, and XOR.

Exercise III.31 Use Toffoli gates to implement FAN-OUT. FAN-OUT would seem to violate the No-cloning Theorem, but it doesn't. Explain why.

Exercise III.32 Design a quantum circuit to transform $|000\rangle$ into the entangled state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

Exercise III.33 Show that $|+\rangle, |-\rangle$ is an ON basis.

Exercise III.34 Prove:

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \\ |1\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \end{aligned}$$

Exercise III.35 What are the possible outcomes (probabilities and resulting states) of measuring $a|+\rangle + b|-\rangle$ in the *computational basis* (of course, $|a|^2 + |b|^2 = 1$)?

Exercise III.36 Prove that $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$.

Exercise III.37 Prove:

$$\begin{aligned} H(a|0\rangle + b|1\rangle) &= a|+\rangle + b|-\rangle, \\ H(a|+\rangle + b|-\rangle) &= a|0\rangle + b|1\rangle. \end{aligned}$$

Exercise III.38 Prove $H = (X + Z)/\sqrt{2}$.

Exercise III.39 Prove Eq. III.18 (p. 118).

Exercise III.40 Show that three successive CNOTs, connected as in Fig. III.11 (p. 117), will swap two qubits.

Exercise III.41 Recall the conditional selection between two operators (C.3, p. 117): $|0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1$. Suppose the control bit is a superposition $|\chi\rangle = a|0\rangle + b|1\rangle$. Show that:

$$(|0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1)|\chi, \psi\rangle = a|0, U_0\psi\rangle + b|1, U_1\psi\rangle.$$

Exercise III.42 Show that the 1-bit full adder (Fig. III.15, p. 119) is correct.

Exercise III.43 Show that the operator U_f is unitary:

$$U_f|x, y\rangle \stackrel{\text{def}}{=} |x, y \oplus f(x)\rangle,$$

Exercise III.44 Verify the remaining superdense encoding transformations in Sec. C.6.a (p. 123).

Exercise III.45 Verify the remaining decoding cases for quantum teleportation Sec. C.6.b (p. 128).

Exercise III.46 Confirm the quantum teleportation circuit in Fig. III.21 (p. 129).

Exercise III.47 Complete the following step from the derivation of the Deutsch-Jozsa algorithm (Sec. D.1, p. 138):

$$H|x\rangle = \sum_{z \in \mathbf{2}} \frac{1}{\sqrt{2}}(-1)^{xz}|z\rangle.$$

Exercise III.48 Show that $\text{CNOT}(H \otimes I) = (I \otimes H)C_Z H^{\otimes 2}$, where C_Z is the controlled-Z gate.

Exercise III.49 Show that the Fourier transform matrix (Eq. III.25, p. 147, Sec. D.3.a) is unitary.

Exercise III.50 Prove the claim on page 165 (Sec. D.4.b) that D is unitary.

Exercise III.51 Prove the claim on page 166 (Sec. D.4.b) that

$$WR'W = \begin{pmatrix} \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \end{pmatrix}.$$