Mathematics for Neural Nets and Especially Field Computation

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Chapter 1

Basic concepts of topology

1.1 Metric spaces

1.1.1 Metric

Definition 1.1.1 ($\mathcal{R}_*$) *$\mathcal{R}_*$ is the set of all nonnegative real numbers, $\mathcal{R}_* = \{x \in \mathcal{R} \mid x \geq 0\}$.*

Definition 1.1.2 (Metric) *A* metric *is a function $d : X \times X \longrightarrow \mathcal{R}_*$ satisfying (for all $x, y, z \in X$):*

Self-identity *$d(x, y) = 0$ if and only if $x = y$.*

Symmetry *$d(x, y) = d(y, x)$.*

Triangle inequality *$d(x, y) + d(y, z) \geq d(x, z)$.*

Remark 1.1.1 *Although these are very natural postulates for a distance measure, they are not sacred, and in particular cases meaningful measures of similarity or dissimilarity may not satisfy them. The triangle inequality is particularly problematic, and a measure that drops this requirement is called a* semimetric. *Moreover, in psychological spaces symmetry may not hold (e.g., x may seem closer to y than y seems to be to x). Finally, the Self-identity property implies that all differences are "measurable" (by the metric). A* pseudo-metric *is function satisfying the weaker condition $d(x, x) = 0$ instead of Self-identity.*

Exercise 1.1.1 *Give an example of a function that is a pseudo-metric but not a metric.*

Exercise 1.1.2 *For a given nondirected graph, define the distance between two vertices to be the number of edges in the minimum path between the vertices. (If there is no path between the vertices, then set the distance to ∞ or to any number greater than the number of vertices.) Is this distance a metric?*

Exercise 1.1.3 *For a given* directed *graph, define the distance from one vertex to another to be the number of edges in the minimum path from the one to the other (i.e. respecting the direction of edges). (If there is no path from one vertex to the other, then set the distance to ∞ or to any number greater than the number of vertices.) Is this distance a metric?*

1.1.2 Examples

The most common metric is the L_2 or Euclidean metric on an n-dimensional vector space:

Definition 1.1.3 (Euclidean Metric) *The L_2 or Euclidean metric on $\mathcal{R}^n$ is defined*

$$L_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

Exercise 1.1.4 *Show that L_2 is a metric.*

The L_p metrics are simple generalizations of the Euclidean metric.

Definition 1.1.4 (L_p Metrics) *For any (real or integral) $p \geq 1$,*

$$L_p(\mathbf{x}, \mathbf{y}) = \sqrt[p]{\sum_{k=1}^{n} |x_k - y_k|^p}.$$

Exercise 1.1.5 *The set of all points $\mathbf{x} \in \mathcal{R}^2$ such that $L_2(\mathbf{x}, \mathbf{y}) = r$ defines a circle of radius r around center $\mathbf{y}$. Thus $L_2(\mathbf{x}, 0) = 1$ defines a unit circle at the origin. Plot, by computer, the (two-dimensional) L_p-circle $L_p(\mathbf{x}, 0) = 1$, for $p = 3, 4, 5$, etc. What conclusions do you draw?*

Exercise 1.1.6 *Show that L_p is a metric for arbitrary $p > 1$. Hint: Expand the pth power by the binomial theorem.*

Consider a special case $p = 1$ of the preceding definition.

Definition 1.1.5 (L_1 metric)

$$L_1(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{n} |x_k - y_k|.$$

Exercise 1.1.7 *Sketch (without use of a computer) a two-dimensional L_1-circle.*

Exercise 1.1.8 *Prove that L_1 is a metric.*

Hamming distance is widely used in coding and information theory as well as in neural nets.

Definition 1.1.6 (Hamming distance) *The* Hamming distance *between two bit-strings of the same length is the number of bits that differ between them.*

Exercise 1.1.9 *Show directly that Hamming distance is a metric*

Exercise 1.1.10 *Show that Hamming distance is equivalent to the L_1 metric.*

Remark 1.1.2 *To design an error-correcting code capable of correcting errors of m or fewer bits, each legal code is surrounded by an L_1-sphere of illegal codes (so the legal codes have a Hamming distance of $2m$). Then, if an illegal code appears in a signal, it can be replaced by the legal code that is nearest (in Hamming distance).*

The L_p definition can also be extended to $p = \infty$ by taking the limit $p \to \infty$.

Definition 1.1.7 (L_∞ metric)

$$L_\infty(\mathbf{x}, \mathbf{y}) = \lim_{p\to\infty} L_p(\mathbf{x}, \mathbf{y}) = \max_{k=1}^{n} |x_k - y_k|.$$

Exercise 1.1.11 *Sketch (without use of a computer) a two-dimensional L_∞-circle.*

Exercise 1.1.12 *Prove that L_∞ is a metric.*

Sometimes the different components of a vector are not equally significant. In these cases *weighted* L_p metrics can be used.

Definition 1.1.8 ($\mathcal{R}_+$) *$\mathcal{R}_+$ is the set of positive real numbers, $\mathcal{R}_+ = \{x \in \mathcal{R} \mid x > 0\}$.*

Definition 1.1.9 (Weighted L_p metric) *Let $\mathbf{w} \in \mathcal{R}_+^n$ be a positive weight vector. Then, for $p \geq 1$,*

$$L_p^{\mathbf{w}}(\mathbf{x}, \mathbf{y}) = \sqrt[p]{\sum_{k=1}^{n} w_k |x_k - y_k|^p}.$$

Exercise 1.1.13 *Sketch a two-dimensional weighted L_2 circle with the weight vector $\mathbf{w} = (1, 2)$. In general, how does a weight vector affect the shape of an L_p circle?*

Exercise 1.1.14 *Why are the elements of $\mathbf{w}$ required to be strictly positive (i.e. $w_k > 0$ for all k)?*

Remark 1.1.3 *The preceding definitions can be extended to $n = \infty$, that is, to infinite dimensional spaces, a subject that will be taken up in Ch. 3.*

1.1.3 Metric space

Definition 1.1.10 (Metric space) *A metric space is a pair (X, d), where X is an arbitrary set (called the* underlying set*) and $d : X \times X \longrightarrow \mathcal{R}_*$ is a metric on the underlying set.*

Remark 1.1.4 *Many different metric spaces can have the same underlying set. For example, $(\mathcal{R}^n, L_p)$, $p = 1, 2, 3, \ldots$ are all different metric spaces with the same underlying set.*

Exercise 1.1.15 *What can you say about the differences between the metric spaces* $(\mathcal{R}, L_p)$ *for* $p \geq 1$*?*

Definition 1.1.11 (Euclidean space $\mathcal{E}^n$)

$$\mathcal{E}^n = (\mathcal{R}^n, L_2).$$

That is, the n-dimensional Euclidean metric space is the space of n-dimensional real vectors with the usual (Euclidean) metric.

Remark 1.1.5 *Sometimes we call* $\mathcal{R}^n$ *a metric space, in which case we mean* $\mathcal{R}^n$ *with its "usual" metric, that is, the Euclidean metric* L_2*.*

1.1.4 Discrete metric space

Definition 1.1.12 (Discrete metric space) *For any set* X*, the* discrete metric *is defined:*

$$\begin{aligned} d_{\mathrm{d}}(x, x) &= 0, \\ d_{\mathrm{d}}(x, y) &= 1, \quad \text{if } x \neq y. \end{aligned}$$

Remark 1.1.6 *With the discrete metric there are no degrees of closeness; so far as the metric is concerned, the points are either identical or they are as different as they can be. The discrete metric is implicit in the usual treatment of alphabets in formal language theory: that symbols of the alphabet are either identical or not; they are not considered close or far on the basis of alphabetic order, shape, sound, ASCII code, or any other basis. Note, however, that the discrete metric is not limited to finite or even countable sets. For example,* $(\mathcal{R}, d_{\mathrm{d}})$ *is the discrete metric space of real numbers, in which there is no sense of a real number being closer or farther to any other real numbers; it is just a set of independent points.*

Exercise 1.1.16 *Show that the discrete metric is in fact a metric.*

1.1.5 Isometry

Definition 1.1.13 (Isometry or isometric mapping) *A function* $f : X \longrightarrow X'$ *is an* isometry *or* isometric mapping *between metric spaces* (X, d) *and* (X', d') *if and only if*

$$d(x, y) = d'[f(x), f(y)],$$

for all $x, y \in X$*. That is, the function* preserves *distances.*

Remark 1.1.7 *Many mappings between metric spaces are not isometric, yet still satisfy some interesting relation between their metrics, e.g.,* $d(x, y) = cd'[f(x), f(y)]$ *or* $d(x, y) = \log d'[f(x), f(y)]$.

Definition 1.1.14 (Isometric spaces) *Two metric spaces are said to be* isometric *if there is an isometric mapping from one to the other.*

Exercise 1.1.17 *Show that isometry is an equivalence relation.*

1.1.6 Function spaces

In many applications, especially in field computation, we deal with *function spaces*, that is, spaces whose "points" are entire functions. Function spaces will be considered in detail in chapters 2 and 3; here we just consider a couple examples. In making a metric space from a set of functions we have to decide on an appropriate notion of distance for the functions. The significance of different metrics can often be understood by thinking of the distance between two functions as being a measure of the error in the representation of one by the other. That is, if f is what we want and g is what we've got, then $d(f, g)$ measures the error in what we've got.

Definition 1.1.15 $\mathcal{C}_\infty[a, b]$ *is the metric space comprising the set of continuous, real-valued functions on* $[a, b]$ *with the* L_∞ *(or* uniform*) metric:*

$$L_\infty(f, g) = \sup\{|f(x) - g(x)| \mid a \leq x \leq b\}.$$

Note that sup *means, roughly, the maximum, so*

$$L_\infty(f, g) = \max_{a \leq x \leq b} |f(x) - g(x)|.$$

Remark 1.1.8 *This metric space is appropriate when we are interested in the maximum difference between two functions (in the worst case, so to speak).*

Definition 1.1.16 $\mathcal{C}_1[a,b]$ *is the metric space comprising the set of continuous, real-valued functions on* $[a,b]$ *with the* L_1 *metric:*

$$L_1(f,g) = \int_a^b |f(x) - g(x)| \mathrm{d}x.$$

Remark 1.1.9 *This metric space is appropriate when we interested in the total (or average) difference between two functions. In other words, we might be satisfied with an error that is, on the average, small, even it is large in some cases.*

Definition 1.1.17 $\mathcal{C}_2[a,b]$ *is the metric space comprising the set of continuous, real-valued functions on* $[a,b]$ *with the* L_2 *metric:*

$$L_2(f,g) = \sqrt{\int_a^b [f(x) - g(x)]^2 \mathrm{d}x}.$$

Remark 1.1.10 *This Euclidean function space is appropriate for many physical problems (in which energy depends quadratically on some other quantity), and is often chosen for mathematical convenience (since its derivative is linear).*

Exercise 1.1.18 *Suppose* $L_2(f,g) < \epsilon$*. Is there any limit on* $|f(x) - g(x)|$ *for a given value of* x*? What does this tell you about a function approximation that is good in an* L_2 *sense? Suppose* $L_\infty(f,g) < \epsilon$*; what does this tell you about* $|f(x) - g(x)|$ *for particular values of* x*?*

1.2 Topology

The theory of metric spaces deals with general models of nearness and distance based on a metric that quantifies distance. Topology generalizes these ideas by dealing with nearness (neighborhoods) in a context that doesn't require a metric (i.e. a quantification of distance).

1.2.1 Open balls

Definition 1.2.1 (Open Ball $B_r(c)$) *The* open ball *with center c and radius $r > 0$ in a metric space (X, d) is defined:*

$$B_r(c) = \{x \in X \mid d(x, c) < r\}.$$

We call $B_r(c)$ an open ball around *c.*

Remark 1.2.1 *Other names for the open ball $B_\epsilon(c)$ are an* open sphere *and an* ϵ-neighborhood.

Remark 1.2.2 (Closed balls and boundaries) *It is sometimes convenient to talk about* closed *balls or spheres around a point, $\{x \in X \mid d(x, c) \leq r\}$. Likewise, we can talk about the* boundary *of an open or closed ball: $\{x \in X \mid d(x, c) = r\}$. Note that a closed ball includes its boundary, whereas an open ball excludes it.*

Remark 1.2.3 *All of the points of an open ball are in its interior or, equivalently, none of its points are on its boundary (and hence it excludes its boundary. To see this, observe that around any point of the ball you can place another open ball that lies entirely within the first. Conversely, observe that for points on the boundary of a closed ball it is not possible to put an open ball around them that lies entirely within the closed ball. Think about this and draw pictures to make sure you understand it at an intuitive level; it will pay off later.*

Exercise 1.2.1 *Prove the preceding remarks.*

Exercise 1.2.2 *What do the open balls $B_r(c)$ look like in the metric space $(\mathcal{R}, L_p)$?*

Exercise 1.2.3 *What do the closed balls look like?*

Exercise 1.2.4 *What are the boundaries of the (open or closed) balls?*

Exercise 1.2.5 *Describe the open balls in any discrete metric space.*

1.2.2 Neighborhoods

Definition 1.2.2 (Neighborhood) *A subset N of a space X is called a* neighborhood *of $x \in X$ if it contains some open ball around x. That is, N is a neighborhood of x if and only if for some $r > 0$, $B_r(x) \subseteq N$.*

Remark 1.2.4 *Notice that a point cannot be on the boundary of any of its neighborhoods; a neighborhood of a point must provide some "space" around that point.*

Exercise 1.2.6 *Show that each open ball around a point is a neighborhood of that point.*

Exercise 1.2.7 *Show that an open ball is a neighborhood of each of its points.*

Exercise 1.2.8 *Give an example of a neighborhood that is not an open ball.*

Exercise 1.2.9 *Show that in a discrete metric space every set containing a point is a neighborhood of that point.*

Exercise 1.2.10 *Show that the union of two neighborhoods of a point is a neighborhood of that point. Extend your result to the union of any finite family (set) of neighborhoods of the point. Does it also extend to infinite families of neighborhoods?*

Exercise 1.2.11 *Show that the intersection of two neighborhoods of a point is a neighborhood of that point. Extend your result to the intersection of any finite family (set) of neighborhoods of the point. Does it also extend to infinite families of neighborhoods? (Hint: Consider the family $\{B_{1/n}(c) \mid n = 1, 2, 3, \ldots\}$ of neighborhoods of c.)*

1.2.3 Open and closed sets

Definition 1.2.3 (Open set) *A subset of a space is called* open *if it is a neighborhood of each of its points.*

Definition 1.2.4 (Closed set) *The complement of an open set is called a* closed set*.*

Remark 1.2.5 *Intuitively, an open set excludes its boundary. Thus, open intervals (a, b) are examples of open sets in $\mathcal{E}$; they exclude their boundaries $\{a, b\}$. Conversely, closed sets include their boundaries, so the closed intervals $[a, b]$ are examples of closed sets in $\mathcal{E}$. Sets that both include and exclude some of their boundary are neither open nor closed; examples are the half-open intervals $[a, b)$ and $(a, b]$ in $\mathcal{E}$.*

Exercise 1.2.12 *Given the foregoing remarks, it may seem that a set could not be both open and closed. However, $\emptyset$ and $\mathcal{R}$ are both open and closed in $\mathcal{E}$. Explain how this can be.*

Exercise 1.2.13 *Show that in a discrete metric space every subset is open and every subset is closed.*

Exercise 1.2.14 *Show that any union (finite or infinite) of open sets is open.*

Exercise 1.2.15 *Show that any* finite *intersection of open sets is open.*

Exercise 1.2.16 *Give an example of an infinite family of open sets whose intersection is not open.*

Exercise 1.2.17 *Prove that any (finite or infinite) intersection of closed sets is closed.*

Exercise 1.2.18 *Prove that any finite union of closed sets is closed.*

Exercise 1.2.19 *Give an example of an infinite union of closed sets that is not closed.*

1.2.4 Closure and interior of a set

The following are informal definitions to help you understand open and closed sets.

Definition 1.2.5 (Interior) *The* interior *of a set is all of its points except its boundary points.*

Example 1.2.1 *For example, in $\mathcal{E}$, (a, b) is the interior of (a, b), $(a, b]$, $[a, b)$ and $[a, b]$.*

Definition 1.2.6 (Closure) *The* closure *of a set is all of its points as well as all of its boundary points.*

Example 1.2.2 *For example, in $\mathcal{E}$, $[a, b]$ is the closure of (a, b), $(a, b]$, $[a, b)$ and $[a, b]$.*

Answer the following questions on the basis of your intuitive understanding of these concepts (i.e., don't give formal proofs).

Exercise 1.2.20 *Is the interior of a set open or closed?*

Exercise 1.2.21 *Is the closure of a set open or closed?*

Exercise 1.2.22 *What is the union of a set and its boundary?*

Exercise 1.2.23 *What is the (set) difference of a set and its boundary?*

Exercise 1.2.24 *What is the intersection of the closure of a set and the closure of its complement.*

1.2.5 Topological spaces

Topological spaces are more general than metric spaces because they allow us to specify the neighborhood properties of a space directly in terms of its open sets without the need of a metric. N.B. Previously the open sets were defined in terms of a metric (by way of open balls); here the open sets will be given as part of the definition of the topology.

Definition 1.2.7 (Toplogical space) *A* topology $(X, \mathcal{T})$ *comprises an underlying set X and a topology $\mathcal{T}$, which is a family of subsets of X (called the* open *subsets of X) satisfying:*

1. *The empty set is open, i.e., $\emptyset \in \mathcal{T}$.*

2. *The whole space is open, i.e.,* $X \in \mathcal{T}$.

3. *The intersection of any finite number of open sets is open. That is, if* $S_1, \ldots, S_n \in \mathcal{T}$ *then* $\bigcap_{k=1}^{n} S_k \in \mathcal{T}$.

4. *The union of any (finite or infinite) collection of open sets is open. That is, if* $\mathcal{S} \subseteq \mathcal{T}$, *then* $\bigcup_{S \in \mathcal{S}} S \in \mathcal{T}$.

Remark 1.2.6 *As before, the* closed *sets are those whose complements are open in the topology. Notice that a set can be both open and closed in a topology; in particular the empty set and the whole space are both open and closed in all topologies. As before, some subsets of a space may be neither open nor closed.*

Remark 1.2.7 *This definition of a topological space specifies the topology through its open sets. The same topology can also be specified through its closed sets, through its neighborhood system, through its interior operator, through its closure operator, and in other ways.*

Remark 1.2.8 *A metric space has a "natural topology" in which the open sets are just those resulting from arbitrary unions of open balls. Alternately, we can take the open sets to be all those subsets for which every point in the subset can be surrounded by an open ball in the subset. Either way, this is called the* metric topology *for the (metric) space.*

Definition 1.2.8 (Discrete topology) *The* discrete topology *for a set* X *is* $\mathcal{T} = \mathcal{P}(X)$, *the set of all subsets of* X.

Exercise 1.2.25 *Show that the metric topology of a discrete metric space is the discrete topology. Conversely, one can define a discrete metric on any discrete topological space.*

Definition 1.2.9 (Subspace) *A topological space* $(X', \mathcal{T}')$ *is called a* subspace *of a topological space* $(X, \mathcal{T})$ *if* $X' \subseteq X$ *and*

$$\mathcal{T}' = \{S \cap X' \mid S \in \mathcal{T}\}.$$

That is the $\mathcal{T}'$ *is the* relativization *of* $\mathcal{T}$ *to the subset, and it is called the* relative topology *on* $X' \subseteq X$.

Exercise 1.2.26 *Show that the relative topology is in fact a topology.*

1.2.6 Bases

Proposition 1.2.1 *A subset of a topological space is open if and only if it is a union of open balls.*

Exercise 1.2.27 *First part of proof: Show that any union of open balls is open. (Trivial)*

Exercise 1.2.28 *Second part of proof: Show that an arbitrary open set U is a union of open balls. Hint: Each point $x \in U$ is surrounded by an open ball $B_\epsilon(x) \subseteq U$.*

Exercise 1.2.29 *Describe in words the open sets in $\mathcal{E}$.*

Definition 1.2.10 (Base) *A family of open sets is called a* base *for a topology if every open set is a (possibly infinite) union of sets in the base. That is, $\mathcal{B} \subseteq \mathcal{T}$ is a base for $(X, \mathcal{T})$ if for every $U \in \mathcal{T}$ there is a $\mathcal{F} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{F}$.*

Remark 1.2.9 *A topology may have many bases, each of which is capable of generating the entire topology. (Analogously, a vector space can have many bases, and each basis is capable of generating the entire vector space. A vector space is generated from a basis by linear operation — vector addition and scalar multiplication — whereas a topology is generated from a base by the union operation.)*

Proposition 1.2.2 *The open balls of a metric space are a base for the metric topology.*

1.3 Convergence and limits

1.3.1 Convergence in metric spaces

Definition 1.3.1 (Convergence in metric spaces) *In a metric space (X, d) a sequence of points $x_1, x_2, \ldots$ is said to* converge to the limit x, *written $x_n \to x$, if and only if $d(x, x_n) \to 0$, that is, $0 = \lim_{n\to\infty} d(x, x_n)$. More precisely, for any $\epsilon > 0$ there is a positive N such that $d(x, x_n) < \epsilon$ for $n > N$. That is,*

$$\forall \epsilon > 0 \; \exists N > 0 \; \forall n > N : d(x, x_n) < \epsilon.$$

Proposition 1.3.1 *In a metric space, the limit, if it exists, is unique (i.e. a sequence cannot converge to two limits). Hence, if* $x_n \to x$ *we can write* $x = \lim_{n\to\infty} x_n$.

Exercise 1.3.1 *Prove this proposition. Hint: Assume the sequence has two limits* $x \neq x'$, *let* $\epsilon = \frac{1}{2}d(x, x')$ *and derive a contradiction.*

Proposition 1.3.2 *In a metric space,* $x_n \to x$ *if and only if, for each* $\epsilon > 0$, x_n *is eventually in* $B_\epsilon(x)$ *(that is, there is a positive* N *such that* $x_n \in B_\epsilon(x)$ *for all* $n > N$*).*

Proposition 1.3.3 *In a metric space,* $x_n \to x$ *if and only if, for each neighborhood* U *of* x, x_n *is eventually in* U.

Exercise 1.3.2 *Prove these propositions.*

Proposition 1.3.4 *In a metric space, the closure of a subset is the set of all limits of sequences in the subset.*

Remark 1.3.1 *Of course, the limits of some of these sequences lie in the interior, but the important point is that we may have sequences lying in the interior with limits on (but not outside) the boundary. This is the primary significance of the boundary. For example, in* $(0, 1) \subset \mathcal{E}$, *we can have sequences converging to 0, 1, or any point inside the interval, but not to any points less than 0 or greater than 1.*

Remark 1.3.2 *To prove the preceding proposition, suppose* $x_n \in U$ *and* $x_n \to x$*; therefore, for any* $\epsilon > 0$, x_n *is eventually in* $B_\epsilon(x)$. *Therefore every such ball intersects* U *(since it contains some of the* x_n, *which are in* U*). Either* x *is in* U *or it's not; if it is, there is nothing to prove. If it's not, then it must be on the boundary of* U, *since we've shown that every* ϵ*-neighborhood of it intersects* U.

Proposition 1.3.5 *The metric topology is determined by the convergent sequences of points.*

Remark 1.3.3 *The convergent sequences define the closures of subsets, and thus the closed subsets, and thus the topology. This observation reveals some of the intimate connections between distance, neighborhoods and convergence that is captured by the notion of a topological space. (The preceding proposition does not hold for arbitrary topological spaces, but only for those that satisfy the "first countability axiom," which will not be discussed here.)*

1.3.2 Convergence in topological spaces

The preceding propositions motivate the following definition.

Definition 1.3.2 (Convergence in topological spaces) *In a topological space a sequence is said to converge to a limit if it is eventually in every neighborhood of that limit. That is, a sequence of points $x_1, x_2, \ldots$* converges to the limit *x, written $x_n \to x$, if and only if for any neighborhood U of x there is a positive N such that $x_n \in U$ for all $n > N$.*

Remark 1.3.4 *In some topological spaces it is possible for a sequence to converge to two different points, $x_n \to x$ and $x_n \to x'$, but $x \neq x'$. Since this behavior is very unintuitive, the remainder of this section will develop an example; it is not, however, necessary for the remainder of the material.*

Definition 1.3.3 (Finite complement topology) *In the* finite complement topology *for a set, the only open sets are those whose complements are finite. Equivalently, the finite subsets are equivalent to the closed sets.*

Exercise 1.3.3 *Show that the finite complement topology is, in fact, a topology.*

Exercise 1.3.4 *Let $(X, \mathcal{T})$ be a topological space in which X is infinite and $\mathcal{T}$ is the finite complement topology. Show that in such a space every neighborhood is infinite and has a finite complement.*

Exercise 1.3.5 *Suppose that in this space $x_n \to x$ and $y_n \to y$. (For example, if $X = \mathcal{R}$, we might take $1 - 1/n \to 1$ and $-1 + 1/n \to -1$.)*

Show that the sequence $x_1, y_1, x_2, y_2, \ldots$ *converges to* both x *and* y. *(For example, the sequence* $\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{4}, -\frac{3}{4}, \ldots$ *converges to* both $+1$ *and* -1.*) Hint: Pick a neighborhood* U *of* x*; we must show* $x_n, y_n, \ldots$ *is eventually in* U. *Since* $x_n \to x$ *we know* x_n *is eventually in* U, *so the problem is to show that* y_n *is also eventually in* U. *Recall now, that* U *has a finite complement.*

Remark 1.3.5 *This peculiar behavior can happen because in this space the neighborhoods are "very large" and in fact have to overlap; so the neighborhoods cannot shrink down (and become nonoverlapping) in the way we expect for convergence. In topological terminology, the space is not* Hausdorff, *which means that any two distinct points have nonoverlapping neighborhoods. Fortunately, metric spaces and most of the other topological spaces we have to deal with are Hausdorff.*

1.4 Continuity

Intuitively, a function is continuous if infinitesmal variations of its input lead to infinitesmal variations of its output (no "cliff effects").

Definition 1.4.1 (Continuous function) *A function* $f : X \longrightarrow X'$ *between (possibly identical) metric spaces* (X, d) *and* (X', d') *is called* continuous at $x \in X$ *if and only if, for each positive* ϵ *there is a* δ *such that* $d'[f(x), f(y)] < \epsilon$ *whenever* $d(x, y) < \delta$. f *is called* continuous *when it is continuous at every* $x \in X$. *More precisely,*

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall y \in X : d(x, y) < \delta \Longrightarrow d'[f(x), f(y)] < \epsilon.$$

Remark 1.4.1 *This is, of course, the standard "*ϵ*-*δ *definition" of continuity, extended to metric spaces. The following propositions extend it to topological spaces.*

Definition 1.4.2 (Image of a set) *If* $f : X \longrightarrow X'$ *and* $S \subseteq X$, *then the* image of S under f *is defined* $f[S] = \{f(x) \mid x \in S\}$.

Proposition 1.4.1 *In a metric space,* f *is continuous at* x *if and only if, for each positive* ϵ *there is a* δ *such that* $f[B_\delta(x)] \subset B_\epsilon(f(x))$.

Proposition 1.4.2 *In a metric space, f is continuous at x if and only if, for each neighborhood V of $f(x)$ there is a neighborhood U of x such that $f[U] \subset V$.*

Remark 1.4.2 *That is, if the variation of the input is limited to U then the variation of the output will be limited to V.*

The preceding motivate the following definition of continuity for topological spaces.

Definition 1.4.3 (Continuous topological map) *A mapping f from a topological space $(X, \mathcal{T})$ to a topological space $(X', \mathcal{T}')$ is called* continuous at $x \in X$ *if and only if, for each neighborhood V of $f(x)$ there is a neighborhood U of x such that $f[U] \subset V$.*

Remark 1.4.3 *There are many equivalent ways of defining continuous maps on topological spaces.*

Definition 1.4.4 *A map is said to be* continuous on a set *if it is continuous at every point in that set.*

1.5 Homeomorphism

Remark 1.5.1 *Suppose that f is a* continuous, one-to-one *mapping of $(X, \mathcal{T})$ into $(X', \mathcal{T}')$. In effect, f continuously embeds X in X', possible "deforming" it in the process, but not "tearing" it. If there is also a continuous, one-to-one function $g : X' \longrightarrow X$, then we see that each space can be continuously embedded in the other. In this sense they are equivalent (ignoring the continuous deformations). These observations motivate the following definitions.*

Definition 1.5.1 (Homeomorphism) *A* homeomorphism *is a continuous, one-to-one, onto mapping with a continuous inverse.*

Remark 1.5.2 *N.B. A <u>homeo</u>morphism is not the same as a <u>homo</u>morphism!*

Definition 1.5.2 (Homeomorphic spaces) *Two spaces are called* homeomorphic *is there is a homeomorphism between them.*

Remark 1.5.3 *Homeomorphic spaces are considered* topologically equivalent*, because topology is most concerned with properties that are preserved by homeomorphisms ("continuous deformations"). This is the reason topology is sometimes called "rubber sheet geometry" and is the source of the old canard about topologists being unable to tell a doughnut from a coffee cup (because they are topologically equivalent). Properties that hold across homeomorphic spaces are called* topological invariants*.*

Remark 1.5.4 *There are many equivalent ways of defining homeomorphism and homeomorphic, which can be found in topology texts. For our purposes it is sufficient to remember that there must be a correspondence between the two sets that is one-to-one and continuous in both directions.*

1.6 Separability

Definition 1.6.1 (Dense) *A subset S of a space X is said to be* dense *in that space if and only if the closure of S is X*

Remark 1.6.1 *Since the closure of S includes, in addition to S, all the limits of sequences in S, we see that if every element of X can be reached by a sequence in S, then S is dense in X. Indeed, in a metric space the closure of S is exactly the set of all limits of sequences in S (Prop. 1.3.4). In other words, S is dense in X if S is sufficient to generate X either in itself or by means of convergent sequences.*

Example 1.6.1 *The set $S = (-1, 0) \cup (0, 1)$ is dense in $[-1, 1]$ since the points -1, 0 and 1 are all limits of sequences in S.*

Example 1.6.2 *The rational numbers are dense in the reals, since every real is the limit of a sequence of rationals (e.g. its decimal approximations).*

Definition 1.6.2 (Separable) *A space is called* separable *if it has a countable, dense subset.*

Example 1.6.3 *The reals are separable because the rationals are a countable, dense subset.*

Remark 1.6.2 *The significance of a space being separable is that it can be generated (via limits) from a countable set, and a countable set can, in principle, be specified by finite strings over a finite alphabet.* (Why?)

1.7 Completeness

1.7.1 Complete metric space

Definition 1.7.1 (Fundamental or Cauchy sequence) *A sequence* $x_1, x_2, \ldots$ *in a metric space* (X, d) *is called* Cauchy *or* fundamental *if* $d(x_m, x_n) \to 0$ *as* $m, n \to \infty$.

Remark 1.7.1 *Notice that Cauchy sequences need not have limits. For example, in the space* Q *of rational numbers with the usual metric* $(d(q, r) = |q - r|)$, *a sequence of rational approximations to* $\sqrt{2}$, *such as 1, 1.4, 1.41, 1.414,..., is Cauchy but does not have a limit in* Q *(since* $\sqrt{2} \notin Q$*).*

Definition 1.7.2 (Complete) *A metric space is* complete *if every Cauchy sequence in the space converges to a point of the space.*

Remark 1.7.2 *In other words, a complete space includes all its limits. Thus, for example, the real numbers with the usual metric are complete, but the rationals are not complete.*

Exercise 1.7.1 *Show that, with the usual metric,* $[0, 1]$ *is complete but* $(0, 1)$ *is not.*

Exercise 1.7.2 *Show that the integers with the usual metric are complete.*

1.7.2 Completion of metric spaces

Definition 1.7.3 (Completion) *A metric space* (X', d') *is a completion of a metric space* (X, d) *if* X' *is complete and* X *is isometric to a subset of* X'.

Remark 1.7.3 *The fact that X is isometric to a subset Y of X' means that the members of X and Y are "essentially the same"; they can be thought of as different names for the same thing. Therefore, since X' is complete, it has limits for all the sequences in Y, and these limits can added to X in order to complete it. For example, if $f : (X, d) \longrightarrow (X', d')$ is an isometric mapping and $x_1, x_2, \ldots$ is a Cauchy sequence in X, then $f(x_1), f(x_2), \ldots$ will be a Cauchy sequence in X'.* (Why?) *Therefore* $\lim f(x_n)$ *exists in X' and we can* adjoin *this to X to provide a limit for the original sequence x_n. (This "adjoining" is a formal trick for adding elements to a set without interfering with the elements already there; in effect they are "colored" differently so that there is no possibility of confusion.) The domain of f can then be extended to include the adjoined element, so that $f(\lim x_n) = \lim f(x_n)$.*

Proposition 1.7.1 (Uniqueness) *The completion of a metric space (X, d) is "essentially unique" in that the subsets corresponding to X in completions must be isometric. That is, if X is isometric to Y in the completion (X', d'), and X is isometric to Z in the completion (X'', d''), then Y and Z are isometric.* (Why?)

Proposition 1.7.2 *Every metric space can be completed.*

Remark 1.7.4 *Sketch of proof: Let (X, d) be a metric space and let (X^*, d^*) be the space of "Cauchy-sequence equivalence-classes" over X. That is, the elements of of X^* are sets of equivalent Cauchy sequences in X, where two Cauchy sequences x_n, y_n are called* equivalent *if $d(x_n, y_n) \to 0$.* (Show this is an equivalence relation.) *The metric on X^* is defined:*

$$d^*(S, T) = \lim_{n\to\infty} d(s_n, t_n),$$

where $s \in S$ and $t \in T$. That is, the distance between two equivalence classes, S and T, of Cauchy sequences is the distance between the limits of any two representatives, s and t, of these classes. *It is necessary to show that this function is well-defined, that is, that is gives the same result no matter what representatives are chosen, and that the function is a metric.*

Exercise 1.7.3 *Show that d^* is well-defined and a metric.*

Remark 1.7.5 *The space* (X^*, d^*) *is complete.* (Try to prove it!) *Furthermore, it is a completion of* (X, d). *To see this, notice that* $x \in X$ *can be embedded in* X^* *by identifying it with the equivalence class of the (stationary or constant) Cauchy sequence* $x, x, x, \ldots$. (Show that stationary sequences are Cauchy.) *Finally, it's necessary to show that the set of equivalence classes of stationary Cauchy sequences is dense in* (X^*, d^*). (Give it a try.)

Remark 1.7.6 *In this way, real numbers (including irrational numbers) can be considered no more than new names for Cauchy sequences of rational numbers. This is the way mathematicians "construct" the real numbers from the rationals. Since rational numbers can be defined as pairs of integers, in this way the reals can be reduced to integers, which is the way Dedekind and other 19th mathematicians endeavored to "arithmetize geometry" (thus reducing the continuum to a discrete formal system).*

Remark 1.7.7 *In the same way, any metric space* (X, d) *can be completed by replacing the elements of* X *by equivalence classes of stationary sequences, and by adjoining to the space the equivalence classes of Cauchy sequences. Further, since all completions are essentially the same (isometric), the construction of equivalence classes of Cauchy sequences is (essentially) the only way to complete a space.*

Proposition 1.7.3 *Every metric space can be considered the dense subset of a complete metric space. (That is, by ignoring as irrelevant the isometry that embeds the original space in its completion.)*

Remark 1.7.8 *Completion, as its name suggests, is the end of the road, at least so far as limits are concerned. Since a complete space includes all the limits of Cauchy sequences, there is no way these sequences can lead to additional elements.*

1.8 Connectedness

1.8.1 Connection

Definition 1.8.1 (Connected space) *A topological space* $(X, \mathcal{T})$ *is* connected *if it has no disjoint, closed proper subsets* $A, B \subset X$ *such* *that $X = A \cup B$.*

Proposition 1.8.1 *Connectedness is a topological invariant.*

Definition 1.8.2 (Connected set) *A subset S of a topological space is connected if it is connected as a subspace of the topological space.*

Example 1.8.1 *Thus, $\mathcal{R}$ and $(0, 1)$ are connected subsets of $\mathcal{E}$ but $[0, 1] \cup [2, 3]$, $(-1, 0) \cup (0, 1)$ and $\{-1, 1\}$ are not.*

Exercise 1.8.1 *Show that a topological space is connected if and only if it has no* open, *disjoint proper subsets whose union is the space. Hint: Review the definition of a closed set in a topology.*

Definition 1.8.3 (Disconnected) *A space or set is called* disconnected *if it is not connected.*

Exercise 1.8.2 *Show examples of various connected and disconnected subsets of $\mathcal{E}^2$ to illustrate the preceding definitions.*

Definition 1.8.4 (Separated sets) *Two nonempty sets in a space are said to be* separated *if the intersection of each with the closure of the other is empty.*

Remark 1.8.1 *That is, two sets are* not separated *if one can be reached by a convergent sequence in the other.*

Exercise 1.8.3 *Give example of various separated and nonseparated pairs of sets in $\mathcal{E}^2$.*

Proposition 1.8.2 *A set is disconnected if and only if it is the union of two (or more) separated sets.*

Proposition 1.8.3 *Let $\mathbf{2} = \{0, 1\}$. A topological space $(X, \mathcal{T})$ is connected if and only if the only continuous maps from X to $\mathbf{2}$ are constant maps.*

Remark 1.8.2 *Sketch of proof: Since* $\{0\}$ *and* $\{1\}$ *are open in* $\mathbf{2}$*, their inverse images* $A = f^{-1}[\{0\}]$ *and* $B = f^{-1}[\{1\}]$ *are disjoint, open sets in* X *for which* $X = A \cup B$*. Therefore* X *is disconnected. Conversely, if* X *is the union of disjoint open sets* A*,* B*, then we can define* $f(x) = 0$ *for* $x \in A$ *and* $f(x) = 1$ *for* $x \in B$*. This function is continuous, since the inverse images of the open sets of* $\mathbf{2}$ *(namely,* $\emptyset$*,* $\{0\}$*,* $\{1\}$*,* $\mathbf{2}$*) are open.*

Remark 1.8.3 *This proposition says that a continuous function cannot map a connected space to two different points. It implies that discrete categories cannot be defined by a continuous function over a connected space. That is, under conditions of continuity, exact categorization is impossible.*

Exercise 1.8.4 *Show that any two disjoint sets in a discrete topology are separated.*

Definition 1.8.5 (Connected component) *A* (connected) component *of a set is a maximal connected subset of it. The* component of a point *in the space is the (unique) component that contains it.*

Exercise 1.8.5 *Why is the component of a point unique?*

Remark 1.8.4 *A connected space has one component.*

Proposition 1.8.4 *The components of a discrete topology are singleton sets.*

1.8.2 Continua and discontinua

Different topologists define "continuum" differently, but all require a continuum to be connected. Hausdorff (*Set Theory*, 173) calls any closed, connected set a *continuum* (an open, connected set being called a *domain*). More generally (e.g. MT 264) a continuum is required to be *compact* (a concept I have avoided in this course) as well as connected. Some authors (e.g. EM 81C, MEGT 158) further require a continuum to have more than one point. For our purposes, Hausdorff's definition is adequate:

Definition 1.8.6 (Continuum) *A* continuum *is a closed, connected set.*

Definition 1.8.7 (Totally disconnected) *A set is* totally disconnected *if its components are singleton sets.*

Remark 1.8.5 *A discrete topology is totally disconnected.*

1.8.3 Path connection

There is another, slightly more restricted definition of connection, which is often more useful.

Definition 1.8.8 (Path or arc) *A continuous function* $p : [0,1] \longrightarrow X$ *is called a* path *or* arc *in the space* $(X, \mathcal{T})$*. The* initial point *of the path is* $x_{\mathrm{i}} = p(0)$ *and the* terminal point *of the path is* $x_{\mathrm{t}} = p(1)$*. The path is said to* join *or* connect x_{i} *to* x_{t}

Remark 1.8.6 *Notice that a path may intersect itself any number of times.*

Remark 1.8.7 *One natural interpretation of a path is as a* continuous trajectory through a continuum state space.

Remark 1.8.8 *A path is a homeomorphic image of* $[0,1]$ *with the usual metric. Indeed, this is a common definition of a path.*

Definition 1.8.9 (Path-connected) *A space is called* path-connected *(*arc-connected, pathwise-connected*, etc.) if from any point in the space there is a path to any other point in the space. A set is path-connected if it is path-connected as a subspace (i.e. in the relative topology).*

Remark 1.8.9 *Therefore, in a path-connected space, any point is reachable from any other point. Alternately, any point can be continuously transformed into any other point of the space.*

Remark 1.8.10 Path-connected components *can be defined in the same way as (connected) components.*

Remark 1.8.11 *The preceding definitions of connection are both quite intuitive, and one might wonder if they are equivalent; they are not. Consider the following set: $Y \cup C$, where Y is the y-axis between -1 and 1, $Y = \{(0, y) \mid -1 \leq y \leq 1\}$, and C is the* graph *of the positive cosine-reciprocal function, $C = \{(x, y) \mid y = \cos(1/x), x > 0\}$. Notice that C oscillates more and more rapidly as it approaches the y-axis, but it never actually touches it. Therefore, $Y \cup C$ is not path connected, since there are no paths between points in Y and points in C. On the other hand, it is connected, as we can see by noting that C is approaching Y, and therefore the closure of C is not disjoint from Y.*

However,

Proposition 1.8.5 *Every path-connected set is connected.*

Exercise 1.8.6 *Prove this proposition. Hint: Suppose the set is the union of two disjoint subsets. You know there is a path from any point in one to any point in the other. Show from this that the subsets cannot be separated.*

1.8.4 Processes

We need to be able to talk about continuous processes in topological spaces, but we do not have the usual mechanisms for defining them, such as differential equations.

Definition 1.8.10 (Process) *A* process *on a topological space S (the* state space*) is a continuous function $f : S \times \mathcal{R} \longrightarrow S$ satisfying the* group properties*:*

$$\begin{aligned} f(s, 0) &= s, \\ f[f(s, t), u] &= f(s, t + u). \end{aligned}$$

The intuitive meaning of $f(s, t)$ is the state of the system t time units after it is in state s. The two preceding equations follow from this intuitive definition.

Exercise 1.8.7 *Explain in words why you would expect each of these equations to be true.*

Remark 1.8.12 *Notice that for any $t \neq 0$ a process f defines a path from s to $f(s,t)$.*

Exercise 1.8.8 *Prove the preceding remark.*

Definition 1.8.11 (Trajectory) *If there is some $t > 0$ such that $f(s_{\mathrm{i}}, t) = s_{\mathrm{f}}$, then we may speak of a* trajectory *from s_{i} to s_{f}. Notice that by this definition of trajectory and process, a trajectory cannot interset itself (i.e. go in two different directions from a the point of intersection).*

Exercise 1.8.9 *Explain why.*

1.9 References

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