Chapter 2

The Continuous and the Discrete

2.1 Word Magic

> All words are spiritual, nothing is more spiritual than words.
>
> — Walt Whitman

> He who shall duly consider these matters will find that there is a certain bewitchery or fascination in words, which makes them operate with a force beyond what we can naturally give account of.
>
> — Robert South

We can never be wholly free of our background of assumptions, but we can become more aware of it, and in this way expose it to change. Although I claimed in the last chapter, and will show in detail in this chapter, that the traditional theory of knowledge grows out of attitudes prevalent in early Greek philosophy, in fact they are grounded in a reverence and awe of language that is common to all cultures.

For if we look, especially in less scientific cultures, we find magic power attributed to words.[1] It's well known that in many societies everyone has

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[1] Sources for this section are Cornford (*FRP*), Englefield (*Lang.*, Ch. 11), Frazer (*GB*, pp. 244–262), Frazer (*NGB*, pp. 235–246), Ogden & Richards (*MoM*, Ch. 2).

a secret name that's known only to one's closest family, because it's believed that anyone who knows an individual's true name (the secret name) has power over that individual. Indeed, a person's name and soul are effectively identical. Likewise, many religions believe that knowing the name of a god grants some control over that god, or that the name of a god should not be spoken out loud (hence the "unspeakable name" represented by the tetragrammaton 'YHWH' was pronounced *Adonai*, "Lord").

This magical power is not limited to personal names, for, as Cornford (*FRP*, p. 141) says,

> To classify things is to name them, and the name of a thing, or of a group of things, is its soul; to know their name is to have power over their soul.

You may wonder how words came to be invested with such power. One theory is that the earliest forms of communication were imperative, and that many magical procedures had their origin in verbal and nonverbal commands: spoken orders (spells), gestures, facial expressions (the "evil eye"), pantomime (ritual dances), etc. (Englefield, *Lang.*, pp. 124–127). However, in the case of word magic a more direct source is apparent, for in many cases words do in fact operate directly to produce an effect. Verbal formulas of this kind are called *performatives* (Austin, *PP*, Ch. 10) because they perform some action. A familiar example of a performative is the formula "I now pronounce you husband and wife." The mere uttering of this phrase by an authorized person (legally or religiously ordained) in an appropriate ceremony is sufficient to make the marriage a *fact*. This is true in general: performatives do not ask or even command that something be done; they *do it*.

Performatives often begin with formulas such as "I hereby . . . " or "By the power vested in me . . . " that signal the special nature of the utterance. They have causal efficacy only if uttered in the appropriate circumstances (e.g., a marriage, graduation or other ceremony) by someone duly authorized. Some performatives require no special authority, such as "I apologize" or "I promise," but even in these cases they may not be efficacious if uttered by a young child, by a mentally incompetent adult, or under duress, etc.

The connection with word magic should be clear. In all societies, but especially in authoritarian ones, many states of affairs can be created by an authorized individual uttering the appropriate verbal formula. Marriage, banishment, official office, death, kinship, identity, possession, access to food or shelter — all may be granted or refused by speaking the right words in

the right way. Is it any wonder that the power came to be attributed to the words themselves rather than to the social context of their use?

> Language, that stupendous product of the collective mind, is a duplicate, a shadow-soul, of the whole structure of reality; it is the most effective and comprehensive tool of human power, for nothing, whether human or superhuman, is beyond its reach. (Cornford, *FRP*, p. 141)

Thus it is hardly surprising that we should find in the earliest philosophy an attempt to capture the world by verbal formulas.

2.2 Pythagoras: Rationality & the Limited

> What is the wisest thing? Number; but second, the man who assigned names to things.
>
> — Pythagoras (attributed in Iamblichus, *Vita Pythagorae* 82; DK 58C4)

> [The Pythagoreans] took numbers to be the whole of reality, the elements of numbers to be the elements of all existing things, and the whole heaven to be a musical scale and a number.
>
> — Aristotle, *Metaphysics* 1.5.985b23 (DK 58B4)

> And indeed all the things that are known have number; for it is not possible for anything to be thought of or known without this.
>
> — Philolaus (DK 44B4)

> There is divinity in odd numbers, either in nativity, chance or death.
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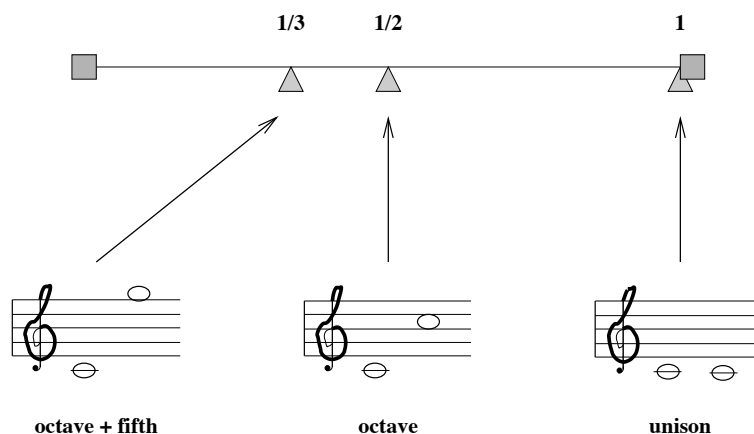> — Shakespeare, *The Merry Wives of Windsor*, 5.1.2

Figure 2.1: Intervals and ratios of lengths. If the string is divided in half (and all other factors are kept constant), then the string sounds an octave higher. If the string is divided in thirds, then it sounds an octave and a fifth higher.

2.2.1   Discovery of the Musical Scale

The early Pythagoreans — perhaps Pythagoras himself (572–497 BCE) — discovered the relationship between musical intervals and ratios.[2] They discovered that strings divided in the ratio 1:2 sounded consonant, producing the interval that we call an octave. Similarly, a ratio of 2:3 produces the interval of a perfect fifth, and 3:4 produces a perfect fourth (Figs. 2.1 and 2.2). It might seem that this discovery's main significance is in music, but in fact it became a paradigm for most later science, logic, mathematics and philosophy. This claim will take some justification, and that is the aim of this section.   To understand the significance of this discovery, it's important to observe that tuning a musical instrument is a skill that requires some training and expertise. It is not easy to describe how the instrument sounds when it's in tune. Rather, the teacher must show the students, who must learn to recognize the difference with their own ears. In this sense, tuning is

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[2]Much of this discussion of the Pythagoreans is drawn from Burnet (*GPI*, §§ 28–38 and §§ 69–75). Another good source is Kirk, Raven & Schofield (*Presoc.*), although the first edition (Kirk & Raven, 1957) is more outspoken in its opinions. Greek philosophical terms are discussed in Peters (*GPT*). I have also used Liddell, Scott & Jones (*LSJ*), and occasionally Donnegan (*Lex.*). Other sources for the Pythagoreans are Burkert (*LSAP*), Sinnige (*M&I*) and Maziarz & Greenwood (*GMP*).
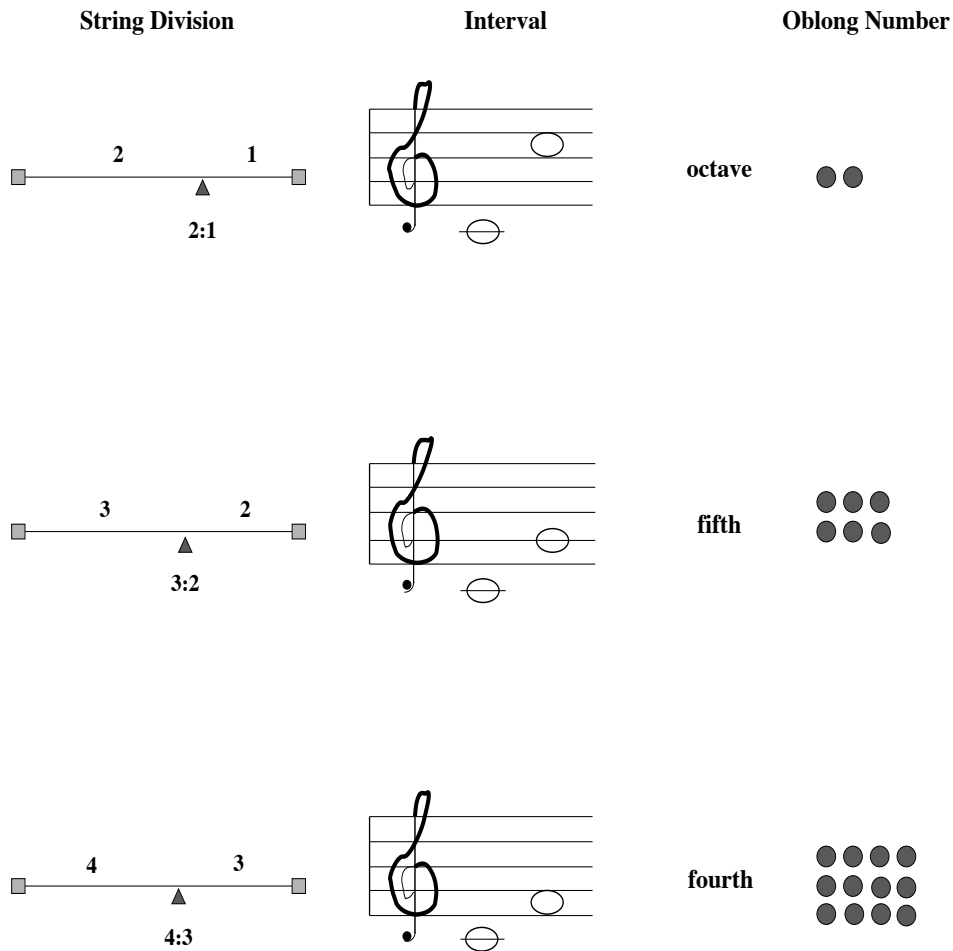
Figure 2.2: Musical intervals based on ratios. If the string is divided in the ration 1:1, then its halves sound the same pitch, called *perfect unison*. The "oblong numbers" (ratios of the form $n + 1 : n$) determine progessively more dissonant intervals. If the string is divided in the ratio 2:1, then its parts sound in the interval of an *octave*, which is the most consonant interval after unison. The ratio 3:2 produces a *perfect fifth*; 4:3 a perfect fourth; 5:4 a major third, 6:5 a minor third; and so on.

apparently inexplicable; that is, we cannot explain it in words.

The accomplishment of Pythagoras was to show that tuning *is* explicable. Specifically, he showed that being in tune is equivalent to satisfying certain ratios. The measurement of these ratios, in turn, is a simple procedure that does not require a "well-trained ear." As Maziarz & Greenwood (*GMP*, p. 43) say,

> Intervals between sounds perceptible only to the fine ears of expert musicians, which could be neither explained to others nor referred to definite causes, were now reduced to clear and fixed numerical relations.

The impact of this discovery on Greek thought was profound. Burnet (*GPI*, p. 56) claims that the concordant intervals

> yield the conception of 'form' as correlative to 'matter', and the form is always in some sense a Mean. This is the central doctrine of all Greek philosophy to the end, and it is not too much to say that it is henceforth dominated by the idea of [harmonia] or the tuning of a string.

(Note that Greek *harmonia* (ἀρμονία) doesn't mean harmony in the modern sense: "the word 'harmony' ... means in the Greek language, first, 'tuning,' and then 'scale' " (Burnet, *GPI*, p. 45).)

In modern terms, what the Pythagoreans accomplished was to reduce a kind of *expertise* (tuning) to a simple *rule* (a ratio). Thus it is both an example of the reduction of an expert judgement to computation, and an example of embodying a phenomenon of nature in a mathematical law. Next we'll discover why it was of crucial importance to the Greeks that the rules was expressed as a ratio.

2.2.2   The Rational

Occasionally, but especially in this chapter, we will consider the origin of some word or group of words. We make these etymological forays for several reasons. First, the histories of these words are part of the archaeology of the theory of knowledge; they exhibit ancient habits of thought from which we derive our own habits. Second, since this book is concerned with knowledge representation, and especially with the role in it of concepts and language

(recall its title, <u>*Word and Flux*</u>), therefore these historical data become examples of the very phenomena of interest. They show us the complexity, in actual use, of the meaning of certain key words, and how the constellation of meanings of such a word may influence the ways we think about the world. Thus these etymological discussions should be read both as pertaining to the history of the theory of knowledge and as illustrating the interplay of language and cognition.

We begin by considering the way the Greek word *logos* (λόγος) was used in Pythagoras' time, which will help us appreciate the significance to the ancient Greeks of the reduction of a natural phenomenon to ratios. This word ultimately derives from the verb *to say* (*legō*), and so the most basic meanings of *logos* relate to saying. By the time of Pythagoras, *logos* could mean word, language, talk and thought. In a more extended sense it could refer to verbal accounts of things, such as reasons, explanations, principles, meanings and causes. Finally, *logos* could mean a ratio or calculation, which is an explanation or reason in the mathematical domain. From *logos* we of course get such terms as *logic* and *logical* as well as the *-ology* that ends the names of many sciences.

It is important to realize that for the ancient Greeks these meanings formed a whole. Thus, that which had a *logos* was simultaneously that which was reasonable, explainable, principled, meaningful, reducible to causes, thinkable and sayable. Conversely, *alogos* came to mean irrational, inexplicable, unprincipled, meaningless, causeless, incomprehensible and unspeakable. (See also p. 29.)

It's easy to see how the Greeks would view the discovery of the musical scale as a triumph of reason over unintelligibility, as indeed it was. The Pythagoreans believed that just as tuning had been reduced to ratios, so eventually all phenomena would be reduced. Hence their claim: "Everything is number."[3]

We need to make one more linguistic observation. The Latin word *ratio* was used with a similar constellation of meanings to the Greek *logos* (in part

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[3]There is disagreement about whether the Pythagoreans said everything *is* number or everything *is like* number; indeed their position may have changed over time. Aristotle states quite clearly however that the Pythagoreans thought numbers were the actual material constituents of things, and that in this they differed from the Platonists (Aris., *Met.* 1.6.6.987b28–29; see also quotation on p. 15). It is noteworthy that *harmony* and *arithmetic* both derive from the same Indo-European root *ar-*, meaning 'to join together' (*AHD*).

because it was used to translate *logos*). In extended senses *ratio* meant a reckoning, account, computation, calculation, list, catalog, relation with or reference to, plan or procedure, principle, reason, ground, method, order, rule, theory, system, knowledge, opinion, or ratio. *Ratio* is of course the source of our word *rational* and its derivatives. It is thus no coincidence that in English *rational* can mean both *expressible as a ratio* and *intelligible*. To the ancients, what was intelligible was what was expressible in words, and a numerical ratio was the paragon of such expressions. Thus, to the Pythagoreans, the rational — in the sense of intelligible — was identical with the rational — in the sense of reducible to ratios.

The connection between what we may call *mathematical* rationality and *epistemological* rationality may seem no more than a historical curiosity, but we will see that over the centuries the two notions have influenced each other in mathematics, logic, philosophy and computer science.

We turn next to the Pythagorean theory of numbers. This will help us understand their idea of ratio (logos). More importantly, however, we will see that it is the ultimate root of formal logic, some critical issues in the foundations of mathematics, the theory of computability, and knowledge representation languages.

The Pythagoreans represented numbers by pebbles, for example, ●, ●●, ●● ●. By placing these pebbles in various arrangements they were able to demonstrate (but not prove in the modern sense) a number of elementary properties of numbers. For example, the triangular numbers can be arranged into an equilateral triangle, which shows that each triangular number is the sum of consecutive integers (Fig. 2.3):

$$
\begin{aligned}
1 &= 1, \\
3 &= 1 + 2, \\
6 &= 1 + 2 + 3, \\
10 &= 1 + 2 + 3 + 4, \\
&\text{etc.}
\end{aligned}
$$

(The Pythagoreans also recognized square, pentagonal, hexagonal numbers, etc.[4]) Similarly, the Pythagoreans were able to prove that the square numbers are the sums of consecutive odd numbers (Fig. 2.4). Notice that if the shape of a *gnomon* (carpenter's square, or rule) is drawn in the figures, then the

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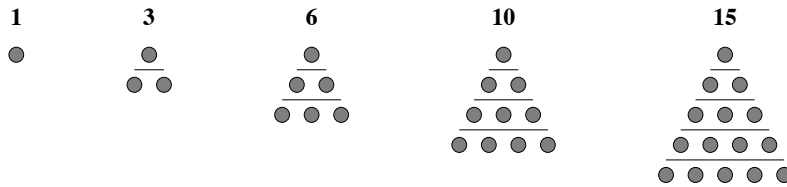[4]See Nicomachus, *Intro. to Arith.* 2.8–12 (Cohen & Drabkin, *SBGS*, pp. 7–9).

Figure 2.3: Triangular numbers. Applying the "rule" shows that consecutive triangular numbers are the partial sums of the natural numbers, 1, $1+2$, $1+2+3$, $1+2+3+4$, ...
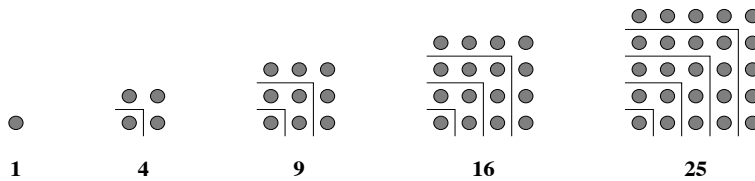

Figure 2.4: Square numbers. Applying the "rule" shows that consecutive square numbers are the partial sums of the odd numbers, 1, $1+3$, $1+3+5$, ...

squares can be seen to be the sums of the odds: $1 = 1$, $4 = 1+3$, $9 = 1+3+5$, and so forth. The oblong numbers can be arranged in figures in which one side exceeds the other by one unit. The oblong numbers are the sums of consecutive even numbers, as can be seen by applying the *gnomon* (Fig. 2.5).

Like the English word 'rule', the ancient Greek *gnōmōn* ($\gamma\nu\acute{\omega}\mu\omega\nu$) could refer either to an instrument that makes something known (such as a carpenter's square, a ruler, a quadrant, or the needle of a sundial), or more generally
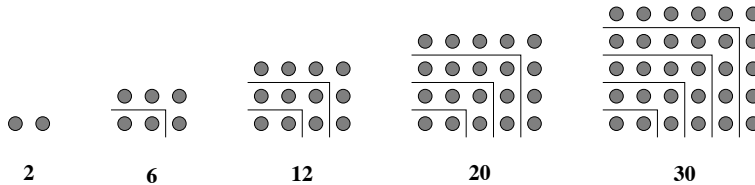

Figure 2.5: Oblong numbers. Application of the "rule" shows that consecutive oblong numbers are the partial sums of the even numbers, 2, $2+4$, $2+4+6$, ...

to a rule to be followed or to one who knows, judges or interprets. It is related to some of the words meaning *to know* (*gignōskō*), knowledge (*gnōsis*) and a rule or principle (*gnōmē*). Thus, concretely, the series of square and oblong numbers are generated by applying the shape of the carpenter's square, but more abstractly by applying an intelligible (*gnōrimon*) rule.[5]

The Pythagoreans were very impressed by the fact that these families of structures were generated by the *recursive* application of a single rule (Sinnige, *M&I*, p. 70). Recursive generation is still valued in science: sentences in the formal grammars used by linguists, logicians and computer scientists, definitions in mathematics, knowledge representation structures in AI and cognitive science — all of these make use of the recursive application of a finite number of rules to a finite number of terms.

In logic, AI and cognitive science, we often refer to a formal pattern as a *schema*, and it is no coincidence that this is the word ($\sigma\chi\hat{\eta}\mu\alpha$) the Pythagoreans used for the shape in which the pebbles were arranged. The corresponding Latin word, *figura*, is the origin of our word *figure*, and it is due to Pythagorean *figured numbers* that we still call numbers *figures* and refer to calculation as *figuring*. Both the Greek and Latin terms refer to the patterns or arrangements of things. Another term used to refer to the arrangement was Greek *eidos* ($\epsilon\hat{\iota}\delta o\varsigma$), which comes from *to see*, and means appearance, aspect, form, figure, kind and so forth. Significantly, it was also used to refer to musical scales. A related word, *idea* ($\iota\delta\acute{\epsilon}\alpha$), is the origin of our word *idea* and is one of the terms Plato used to refer to his *forms* (Section 2.4). The latter is just the English derivative of the Latin *forma*, which has a similar meaning to the Greek *schema*. It is the basis of our notion of a *formal* system.[6] But also we see the roots of an assumption that *ideas* are formal structures, and hence that intelligence may be reduced to a formal system.[7]

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[5]See Donnegan (*Lex.*), *LSJ*, Sinnige (*M&I*, pp. 70–75) and Peters (*GPT*). The Greeks got the *gnomon* — the needle of the sundial — from the Babylonians, who also arranged pebbles in right triangles for calculating their sides (Kirk, Raven & Schofield, *Presoc.*, pp. 83, 103, 335; see Neugebauer, *ESA*, Ch. II).

[6]See Burnet (*GPI*, pp. 49–53) and Taylor (*VS*, Ch. 5), as well as pp. 16 and 44.

[7]Pythagorean representation of numbers in *figures* may have been suggested to them by the constellations, and they probably knew that the Babylonians distinguished two aspects of a constellation: the number of stars in it and their arrangement (Maziarz & Greenwood, *GMP*, p. 13). The Babylonian view may have suggested to the Pythagoreans a distinction between the substance and the form of a thing, a characteristic feature of later Greek philosophy.

Figure 2.6: Termini or Boundary Markers.

The Pythagoreans called the pebbles in their figured numbers *boundary stones*, and called the spaces that they defined *fields*. However, the Greek word for these stones (ὅρος, *horos*) and its Latin equivalent, *terminus*, have a spread of meanings, including landmark, stone tablet, boundary, and, more abstractly, limit, standard, measure, aim, goal, rule and definition. These all connote definition or delimitation. This constellation of meanings is still with us in our *term*. We refer to a term in logic and mathematics, a technical term, to run to term, a school term, a prison term, terms of surrender or agreement, and speak of coming to terms with, and being on equal or good terms with. These all have connotations of mark, limit, measure or goal.

It is of course reasonable that in early agricultural societies the marking out of fields by boundary stones is fundamental to the structure of the society. They provided a definitive basis for resolving land disputes, and it is easy to imagine their becoming the principal metaphor for anything that is defining, delimiting, or conducive of order. As evidence of the importance of boundary markers, we find that in ancient Rome: "Offenses against the gods included murder, the slaying of a parent, incest, the selling of one's wife, the swearing of false oaths, and the moving of boundary stones, this last being a particular affront to the god Terminus" (Humez & Humez, *ABC*, p. 123). (See Fig. 2.6.) At one time, anyone pulling up such a stone could be killed with impunity

and without the killer becoming defiled. The importance of Terminus is illustrated by the story that he was the only god that refused to give way to Jupiter when the latter came to reside on the Capitol. *Termini* (boundary stones, terms) were considered statues of the god and so were crowned with garlands and honored with sacrifices. Terminus was also celebrated in year-end festivals:[8]

> The simple neighbors meet and hold a feast, and sing thy praises, holy Terminus; thou dost set bounds to peoples and cities and vast kingdoms; without thee every field would be a root of wrangling. (Ovid, *Fasti* 2.657–660)

Finally, recall also that *geometry* means *the measurement of land* and that we are told by Herodotus (*History* 2.109) that it had its origins in Egyptian surveying.[9] Thus both number theory and geometry have their origins in the dividing of continuous land. (See also Section 2.2.3 and Cohen & Drabkin, *SBGS*, p. 34.)

Significantly, the words *horos* and *terminus* were used to refer to the terms of a proposition or of a ratio. And there you have it. For the ancients *terms* were tokens that, by a recursive *rule*, could be arranged into *forms*, *figures* and *schemas*, and which thereby put knowledge in *rational* (or *logical*) form. This became the dominant root metaphor for knowledge for the next 2500 years.

It is well known that in ancient times small pebbles were used for calculation, for voting, and in various games. Indeed, it now seems that writing itself may have had its origin in the use of clay tokens for accounting (Schmandt-Besserat, ARS, EPW). In early Neolithic times, eleven thousand years ago, Mesopotamian merchants began to enclose tokens of various standard shapes in a clay envelope to indicate the contents of a shipment (i.e., a bill of lading). However, since the contents of the envelopes could not be

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[8]See Guirand (*NLEM*, p. 214); Lemprière (*Cl. Dict.*, p. 615); Mollett (*Dict.*, pp. 315–316); Nettleship & Sandys (*Dict.*, pp. 620–621); *OCD*, p. 1045; Yonah & Shatzma (*Enc.*, p. 450).

[9]The impetus for the development of practical geometry may been the high population density in the Mediterranean region (Cornford, *FRP*, p. 142). Ancient tradition (Herodotus, Aristotle, Eudemos) held that geometry was brought to Greece by Thales (624–550 BCE). Although both the Egyptians and the Babylonians knew some practical geometry, its development as a logically structured science seems to have been initiated by Greeks, perhaps Thales himself (Ronan, *Science*, p. 68).

checked without breaking them open (which would be done only by the final recipient of the shipment), it was convenient to impress the shapes of the tokens on the outside of the envelope. Eventually, towards the end of the fourth millennium BCE, the enclosed tokens were omitted and their shapes were simply impressed on tablets, the predecessor of an ideographic writing system.

The *terms* of the Pythagoreans are of course another such use. The Latin word for such a token is *calculus*, and it is from the manipulation of calculi that we get our word *calculate*. We still use the word *calculus* for any game-like system in which *terms* are arranged in *schemas* and manipulated according to *formal* rules. The coincidence in terminology is not accidental, as we will see.

2.2.3 The Definite and the Indefinite

There is another issue in Pythagoreanism that we must discuss, for it sets the tone for much of Greek philosophy, and becomes a central issue in the foundations of mathematics and a motivation for symbolic knowledge representation in AI and cognitive science. It is related to the issue of boundary marking that we have already seen. The Greeks were uncomfortable when a continuum was not divided into discrete parcels by delimiting terms.

The root, again, is the notion of a boundary, limit or end ($\pi \acute{\epsilon} \rho \alpha \varsigma$, peras), but the more important term is *apeiros* ($\acute{\alpha} \pi \epsilon \iota \rho o \varsigma$), which is usually translated *infinite*.[10] More precisely it means *without internal or external limit*:

> Thus, in the context of a pre-Socratic philosopheme, and even still in Plato, *apeiron*, when translated into modern idiom, may have to be rendered by: infinite, illimited, unbounded, immense, vast, indefinite, undetermined; even by: undefinable, undifferentiated. (Bochner, Inf., p. 607)

It will be easier for us to understand the issue through Latin terms, since they are cognate to the relevant English words. The verb *finire* means to bound, limit, enclose within limits, restrain, determine, put an end to or conclude. The related noun *finis* means boundary, limit, border, term or territory. Finally, the perfect passive participle of *finire*, which is *finitus*,

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[10]A compact discussion can be found in the entries in (Peters, *GPT*) for *apeiron* and *peras*.

Table 2.1: Pythagorean Table of the Ten Opposites

Limited	Unlimited
Odd	Even
One	Plurality
Right	Left
Male	Female
Resting	Moving
Straight	Curved
Light	Darkness
Good	Bad
Square	Oblong

means that which has been bounded, limited, restrained, ended, etc. This is of course cognate to our word *finite*, but is in fact broader in meaning. It means not just finite as opposed to infinite (endless), but also definite, determinate and limited. It includes the notion not only of a definite end, but also of clear and distinct boundaries and internal divisions. (Recall the importance of boundary markers, p. 23.)

It is hardly surprising that the Greeks considered the finite (in this broad sense) to be intelligible and good, whereas the infinite was chaotic and bad. "For evil belongs to the unlimited, as the Pythagoreans conjectured, and good to the limited" (Aristotle, *Nic. Ethics* 1106b29).

The Pythagorean preference for the definite is also expressed in their *Table of the Ten Opposites* (Table 2.1). These oppositions may be understood as follows.[11]

Limited vs. Unlimited   This is of course the fundamental opposition upon which all the others are based. Indeed, all the others are mixtures, with some of the unlimited entering into the limited.

Odd vs. Even   The opposition is a bit obscure, but perhaps can be understood as follows (Aristotle, *Physica* 203a2). An even number can be divided or analyzed, but an odd number cannot. Therefore, so long as division yields

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[11]Some of these explanations are ancient, but others are modern conjectures.

even numbers, analysis can continue; it is limited or ended by the reaching of an odd number. Therefore the odd numbers are the ultimate limits (or "atoms") of analysis. (We will see later that the notion of analysis stopping at "atoms" is fundamental to traditional epistemology.)

One vs. Plurality   Plurality contains an admixture of the void. For there to be discrete things there must be a principle of separation (void, gap = *chaos* in Greek).[12] In this sense the One is pure, unadulterated by *chaos*. The opposition of the One and the Many is a recurring theme in Western philosophy. The Pythagoreans held that

> the void distinguishes the natures of things, since it is the thing that separates and distinguishes the separate terms in a series. This happens in the first instance in the case of numbers; for the void distinguishes their nature.
> (Aristotle, *Phys.* 4.6.213b24–28)

(This observation is a deep insight into the topological distinction between the continuous and the discrete, as will be explained in volume 2.)

Resting vs. Moving   The resting is *stable*, the moving unstable. The Greeks did not know how to make motion rational; the reduction of motion to ratios was not achieved until Galileo's time. The problem of change was central to all Greek philosophy, and much of it can be seen to be based on the assumption that change is inherently irrational (unintelligible). This is clearest in Plato (p. 43). Aristotle made the understanding of change central to his philosophy, but his theory was qualitative, i.e., he did not succeed in reducing change to ratios. We will see that the mathematical description of change, especially by the calculus, depended on a reconciliation of the rational and the irrational, the discrete and the continuous, the resting and the moving, the straight and the curved — all issues the Pythagoreans had identified.

Straight vs. Curved   Straight lines have a constant (stable, dependable, determinate) direction; curved lines do not; since their direction is always changing, it is indeterminate. Also the length of a curved line is problematic,

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[12]In Greek mythology, as in many others, the origin of the universe of discrete things comes with a separation of the undifferentiated continuum (Robinson, *IEGP*, Ch. 1).

as is the area under a curve. These notions were not clarified until the calculus was developed.

Light vs. Darkness   The simplest explanation here is that in the light we see things clearly and distinctly, whereas in the dark everything is obscure and indeterminate.

Good vs. Bad   The only comment we make here is that it was a persistent theme of ancient Greek culture that the limited was good (cf. "Nothing in excess" on Apollo's temple at Delphi), and the absence of limits was bad (cf., the concept of *hubris*, or "overweaning pride").

Square vs. Oblong   This is an unusual opposition, but it is important for the history of mathematics. The idea seems to be this (Aristotle, *Phys.* 3.4.203a10–15). The series of square numbers maintains a constant ratio of their sides, namely $1/1$. On the other hand, for the oblong numbers this ratio is constantly changing: $1/2$, $2/3$, $3/4$, $4/5$, ... Of course we would say that this series approaches a limit, 1, but the concept of a limit was yet to be invented, and the Greeks had a hard time conceiving of the *limit* of an *unlimited* process. It was two thousand years before this problem was adequately resolved (if in fact it is yet). (See pp. 20 and 30.)

Other Oppositions   The remaining oppositions (male vs. female and right vs. left) are difficult to understand in purely philosophical terms; they probably represent cultural biases of the Pythagoreans. For example, Wheelwright (*Pres.*, pp. 203–204) points out that constancy of direction was considered a masculine characteristic, and changeableness feminine. It must also be mentioned that in addition to their scientific endeavors, Pythagoreanism had a strong mystical component, and that they had many — to our mind — odd doctrines.[13]   On the other hand, the association of the right side with the "propitious, healthy, strong, dexterous," and male, and conversely the left

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[13]Dodds (*GI*, pp. 140–146) has argued persuasively that Pythagoras was a shaman, as were at least two more of the earliest Greek philosophers, Empedocles and Epimenides. It is perhaps no coincidence that we find these shaman-philosophers in late fifth-century and early sixth-century Greece, and that the Greeks' first contact with a culture based on shamanism came in the seventh century, when the Black Sea was opened to Greek trade and colonization. Against this, see Kirk, Raven & Schofield (*Presoc.*, p. 229).

with the "unfavorable, unsound, weak, ... *sinister*" and female, has been a pervasive pattern in the Indo-European cultures (Mallory, *SIE*, p. 140).[14]

2.2.4   The Discovery of the Irrational

The foregoing discussion will perhaps make clear the devastating effect that the discovery of irrational numbers had on the Pythagorean brotherhood. It is possible that Pythagoras himself discovered the property that bears his name, and this led directly to the observation that the sides and hypotenuse of an isosceles right triangle are incommensurable. Let's try to understand this in terms of Pythagorean "figures." A ratio $m : n$ can be represented in a figure as a rectangle with sides m and n. What Pythagoras discovered is that there is *no* formula (arrangement of terms), no matter how big, that can represent this ratio exactly (Fig. 2.7). We can of course approximate it, but the exact ratio is forever beyond our grasp. Thus, although the hypotenuse surely has a length, it cannot be expressed by any (de)finite figure.

The implications of this discovery for the Pythagoreans was that their goal, which was to reduce all of nature to ratios, that is, to produce a rational account of nature, was doomed to failure. They had discovered a phenomenon of nature — in mathematics no less — which was, in their terms, by its nature irrational, and thus forever beyond the grasp of reason. This discovery destroyed the confidence expressed in "Everything is number."

Additional insight into the significance of this discovery on the Pythagorean outlook is provided by the etymology of the words *surd* and *absurd*. The word *surd*, in its mathematical sense of an irrational number, derives from the Latin *surdus* (deaf, inaudible, or insufferable to the ear), which is a translation of the Greek *alogos* (speechless, irrational). On the other hand, *absurd* origi-

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[14] The Pythagorean Table of Opposites may be compared with the ancient (before 400 BCE) Chinese opposition of *yin* and *yang*: "Passive and active principles, respectively, of the universe, or the female, negative force and the male, positive force, always contrasting but complimentary. Yang and yin are expressed in heaven and earth, man and woman, father and son, shine and rain, hardness and softness, good and evil, white and black, upper and lower, great and small, odd number and even number, joy and sorrow, reward and punishment, agreement and opposition, life and death, advance and retreat, love and hate..." (Runes, *Dict.*, s.v. Yin and Yang). Other oppositions associated with yang and yin include light and dark, warm and cold, strong and weak, dynamic and passive, creative and receptive. For the most part the Chinese oppositions agree with the Pythagorean, although it is worth noting that in Taoist thought the yin (feminine) was considered preferable to the yang (masculine) (Schwartz, *WTAC*, p. 203; Laotse, *WoL*, Ch. 28).

1/1 = 1   **3/2 = 1.5**   **7/5 = 1.4**

17/12 = 1.4166. . .,   41/29 = 1.41379. . .,   99/70 = 1.4142857. . .
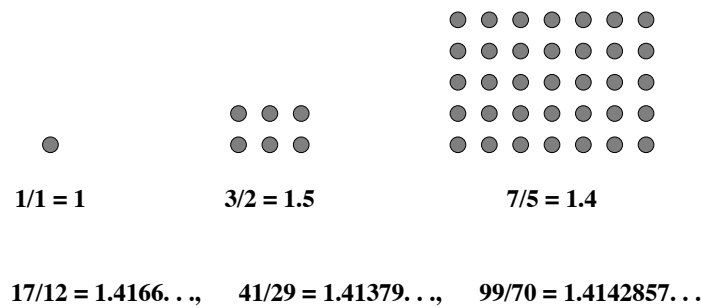
Figure 2.7: Figures Approximating Square Root of 2. The figures come closer and closer to expressing the ratio of the side to the hypotenuse ($\sqrt{2}$), but they never reach it. Hence, the relationship was considered irrational and the process infinite. See also the opposition of square and oblong numbers (p. 28). In this case the "rule" that generates the elements of the series is as follows: the height of the next figure is the sum of the width and height of the previous figure; the width of the next figure is the width plus twice the height of the previous. We have this procedure from Theon of Smyrna (fl. c. 115–140 CE), but it probably goes back to the early Pythagoreans (Heath, *Euclid*, Vol. 2, p. 119; Maziarz & Greenwood, *GMP*, pp. 121–122).
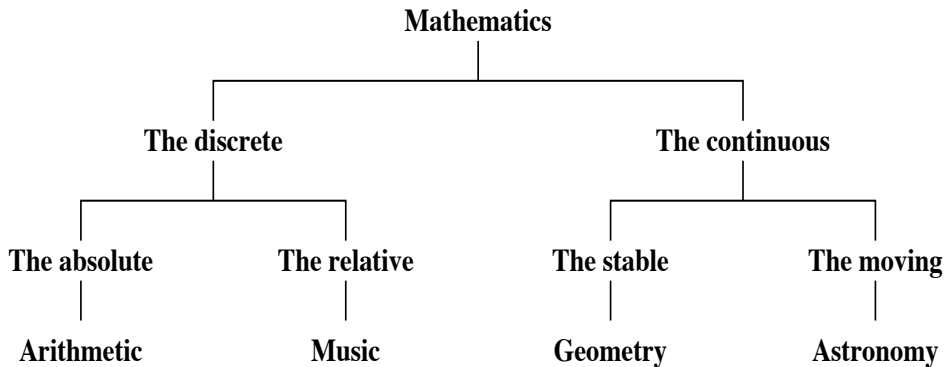
Figure 2.8: Pythagorean Divisions of Mathematics. From late antiquity through the middle ages, the four mathematical sciences were called the *Quadrivium.* Together with the *Trivium* — grammar, logic and rhetoric (which we might call syntax, semantics and pragmatics) — they made up the Seven Liberal Arts of the medieval schools.

nally meant inharmonious, jarring and out of tune (cf. Pythagorean musical theory, p. 20). It comes from *ab* (an intensive), and *surdus.* Thus, to the ancients it was nearly tautological that *surds were absurd.*[15]

2.2.5 Arithmetic vs. Geometry

The discovery of the irrational caused a major setback in mathematics at the end of the fifth century BCE (Maziarz & Greenwood, *GMP*, p. 5), and resulted in a split between arithmetic and geometry that was to last for two thousand years. On the one hand was the Pythagorean arithmetic calculus: the theory of natural numbers seemed like rationality in its truest sense. On the other hand, the demonstrations of the earlier geometers (perhaps Pythagoras himself) seemed convincing. Each of the two sciences, arithmetic and geometry, seemed to yield irrefutable laws, yet they remained unreconciled. As a result, mathematics split into two subdisciplines (Fig. 2.8),[16] and a major research problem in the philosophy of mathematics was born:

Future discussions will center around the 2 Pythagorean oppo-

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[15]*American Heritage Dictionary* and *Oxford English Dictionary*: 'absurd' and 'surd'.

[16]H. W. Turnbull ("The Great Mathematicians," in: Newman, *WM*, p. 85) says we owe to the Pythagoreans this division of mathematics, as well as the word *mathematics* itself.

> sites of the indefinite (continuous) and the finite (discrete). But
> no synthesis of these two principles has yet been found to satisfy
> equally mathematicians and philosophers. (Maziarz & Greenwood,
> *GMP*, p. 65)

Most attempts at a unification of mathematics have tried to reduce geometry to arithmetic, since the calculus-like manipulation of terms in schemas according to formal rules has always seemed more rational. This *arithmetization of geometry* — the attempt to ground geometry in something like Pythagorean number theory — will be discussed in detail below (Chapter 4). Suffice it here to say that the arithmetization of geometry was not accomplished until the nineteenth century (by Dedekind and Weierstrass); the methods lead directly to the theory of computation.

2.3   Zeno: Paradoxes of the Continuous & Discrete

> Zeno's argument, in some form, have afforded grounds for almost all
> the theories of space and time and infinity which have been constructed
> from his day to our own.

> — Bertrand Russell

2.3.1   Importance of the Paradoxes

After the discovery of the irrational in geometry, the Pythagoreans broke into two groups; one concentrated on mathematics, the other had more mystical interests.[17]   Likewise, we shall, for a time, have to follow two parallel paths (they don't rejoin until the nineteenth century). On the one hand we have the history of mathematics trying to reconcile the discrete and the continuous; the only alternative would seem to be to abandon arithmetic or geometry. On the other hand, the second group of Pythagoreans clung to

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[17]Dodds (*GI*, p. 67, n. 68) thinks this "split" is a modern fiction, imposing on the ancient Pythagoreans a modern dichotomy between science and mysticism. For the recency of this dichotomy, see Section 4.2.

the idea that true knowledge is rational, but concluded that the forms are not mathematical (where irrationality is inevitable), but more abstract. This is the path pursued by Socrates and Plato, which we will consider shortly. For now, however, we will follow the mathematical path a little further, and consider Zeno's paradoxes.

Zeno's (c.490 – 430 BCE) aim seems to have been to show that the continuous and the discrete are fundamentally irreconcilable, and in this he was quite successful. "The fact that it took 24 centuries to answer satisfactorily Zeno's arguments proves their fundamental importance in the history of mathematical philosophy" (Maziarz & Greenwood, *GMP*, p. 60). The formal apparatus of limits in modern mathematics makes it is easy to be glib about them, but, considered seriously, they still remain paradoxes. As Hamming (UEM) has said,

> Zeno's paradoxes are still, even after 2,000 years, too fresh in our minds to delude ourselves that we understand all that we wish we did about the relationship between the discrete number system and the continuous real line we want to model.

We'll see that the modern mathematical approach not without its own problems. The fundamental question of the continuous and discrete is: *In what sense in a continuum composed of discrete points?*[18]

2.3.2   Paradoxes of Plurality

As a defense of the thesis of his master, Parmenides, that "everything is one, altogether, changeless" (DK 28B8). Zeno proposed the following paradoxes to show the inconsistency of the idea that things are composed of units, as the Pythagoreans believed:

- The many have no size

- The many have infinite size

- The number of the many is finite and infinite

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[18]An excellent discussion of Zeno's paradoxes can be found in Maziarz & Greenwood (*GMP*, Ch. 6). It is the basis for much of the presentation here. Additional information can be found in Kirk, Raven & Schofield (*Presoc.*, Ch. IX), Robinson (*IEGP*, Ch. 7), Burnet (*GPI*, §§ 63–66) and Sinnige (*M&I*, Ch. IV).
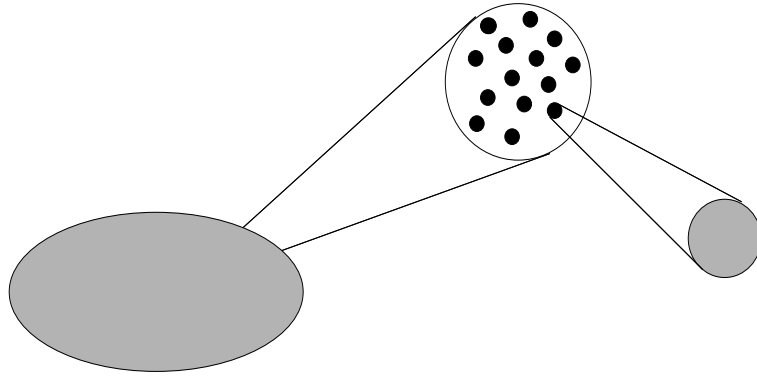
Figure 2.9: The Many Have No Size

We'll consider each in turn.

> *The Many Have No Size:* "If it [the unit] existed, it would have to
> be one. But if it were one, it could have no body. If it had thick-
> ness, it would have parts, and then it would no longer be one."
> (Melissus of Samos, in Simplicius, *Phys.* 109.34; DK 30B9)[19]

The idea seems to be as follows (Fig. 2.9). Suppose that a thing is composed
of units. Then these units must have no size. That is, they must be infinitely
small (infinitesmal), since if they had any size, they would have parts (e.g.
left and right sides). But such a unit doesn't exist at all, "for, having no size,
it could not contribute anything to the size of that to which it was added.
And thus the thing added would be nothing" (Simplicius, *Phys.* 139.5; DK
29B2). See also Robinson (*IEGP*, p. 129).

> *The Many Have Infinite Size:* "If they exist, each must have
> some size and thickness, and one part of it must project beyond
> the other. And the same argument applies to the projecting part;
> for this too will have size, and some part of it will project. Now
> to say this once is the same as saying it forever." (Simplicius,
> *Phys.* 140.34; DK 29B1)

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[19] The abbreviation 'DK' refers to the fragment's "Diels-Kranz number," its position in
Diels & Kranz (*Frag.*). Freeman (*APSP*) provides a translation indexed by DK number,
but Hussey (*Presoc.*, p. 156) claims it is unreliable and recommends instead Kirk, Raven
& Schofield (*Presoc.*), Guthrie (*HGP*), Burnet (*GPI*) or Burnet (*EGP*).
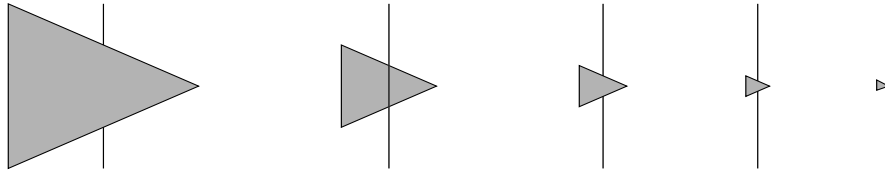
Figure 2.10: The Many Have Infinite Size

The picture may be something like this (Fig. 2.10). If it has size, then it has parts, but these parts also have size. And so we have an infinite number of parts, all with finite size.

> *The Number of the Many is Finite & Infinite:* "If there is a many, there must be just so many — neither more nor less. But if there are just so many, they must be limited in number." That is, a (de)finite number. But, "If there is a many, there must be an infinite number of them. For between existing things there are always others, and between these others still." (Simplicius, *Phys.* 140.27; DK 29B3)

That is, an in(de)finite number. So again we reach a contradiction by assuming that there is a many, that is, that things are composed of discrete units.

2.3.3   Paradoxes of Motion

Zeno's paradoxes of motion can be organized as shown here:

	continuous	discrete
absolute motion	Dichotomy	Arrow
relative motion	Achilles	Stadium

They can be classified in terms of whether they're problems of the continuous or problems of the discrete. The Dichotomy and the Achilles are both problems of the continuous; they show the difficulties that arise when we assume space is infinitely divisible. We are left with an infinite number of pieces, all of finite size. If we think of them as discrete units then they seem to combine to an infinity (Fig. 2.11). The Arrow and the Stadium are
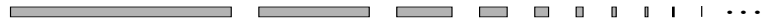
Figure 2.11: Problems of the Continuous. The Dichotomy and the Achilles assume that space and time are infinitely divisible.
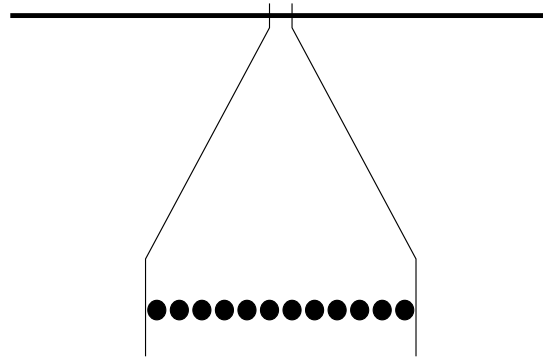
Figure 2.12: Problems of the Discrete. The Arrow and the Stadium assume that space and time are composed of indivisible units.

both problems of the discrete; they show the difficulties that arise when we assume time is composed of discrete moments (Fig. 2.12). I'll discuss each paradox briefly.

> *The Dichotomy:* "There is no motion, because what moves must arrive at the middle of its course before it reaches the end." (Aristotle, *Physics* 239b11)

That is, before we reach the point 1, we must pass through the point 1/2, and before we can do that we must pass through 1/4, and so on (Fig. 2.13). Hence we must pass through an infinity of points — each requiring finite time to reach — in finite time: $1/2 + 1/4 + 1/8 + \cdots$.

> *The Achilles:* "The slower in a race will never be overtaken by the quicker; because the pursuer must first reach the starting point
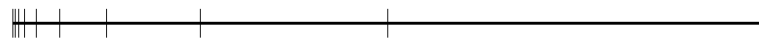
Figure 2.13: The Dichotomy

of the pursued, so that the slower must always be some distance
ahead." (Aristotle, *Physics* 239b14)

Suppose for simplicity that the slower is given a head start of 1 meter, and
that the faster is twice as fast as the slower. By the time the faster has covered
the 1 meter, the slower will have advanced another 1/2 meter. By the time
the faster goes that 1/2 meter, the slower will have gone another 1/4, and
so on. The slower will always be a little ahead of the faster. Of course Zeno
knew as well as we do that the quicker will overtake the slower. The point of
the paradox is to show a contradiction between this common experience and
our theoretical reasoning about continuous motion and infinite divisibility.

> *The Arrow:* "The flying arrow is at rest"; because a thing is at rest
> when occupying its own space at a given time, as the arrow does
> at every instance of its alleged flight. (Maziarz & Greenwood,
> *GMP*, p. 59; cf. Aristotle, *Physics* 239b29, 5)

This is the problem of "instantaneous velocity." Suppose that at a given
indivisible instant the arrow is moving. But if it moves it must be at different
places at different times. But this has divided the instant (into a before and
an after), which contradicts its indivisibility. Thus, in an indivisible instant
the arrow cannot move; it's at rest. But if it's at rest at every instant of
time, then it cannot move at all.

> *The Stadium:* This argument "supposes a number of objects all
> equal with each other in dimensions, forming two equal rows and
> arranged so that one row stretches from one end of a race course
> to the middle of it and the other from the middle to the other
> end. Then if you let the two rows, moving in opposite directions
> but at the same rate, pass each other, Zeno undertakes to show
> that half of the time they take in passing each other is equal to
> the whole." (Aristotle, *Physics* 239b33–240a2)

The argument seems to be this. We have three rows of objects of the same
length. One row is stationary, the other two move in opposite directions.
The initial configuration is shown in Fig. 2.14. Now consider the point in
time when the two moving rows are both aligned with the stationary row
(Fig. 2.15). When this occurs, the first unit in row B will have passed all the
units in row C, but only half the units in row A. But rows A and C are the
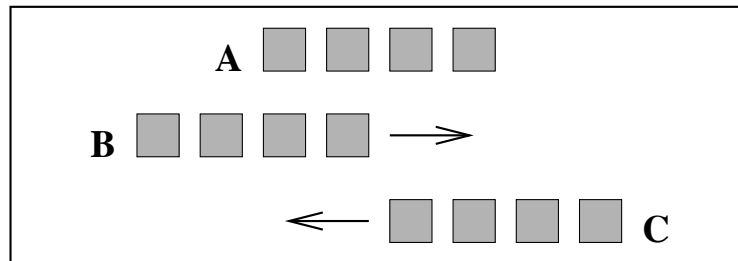
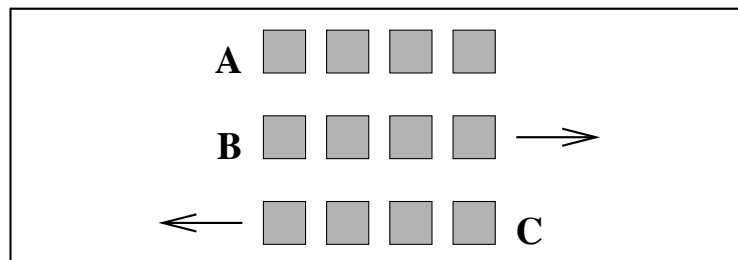Figure 2.14: The Stadium: Initial Configuration

Figure 2.15: The Stadium: Rows Aligned

same length, so in a given period of time it has gone both the distance and half the distance. The contradiction arises from supposing that the units are indivisible. Then, in the time it takes B to pass one unit of A it will pass half of a unit of C, thus contradicting its indivisibility.

2.3.4 Summary

What has Zeno accomplished by these paradoxes? He has shown that if you assume that space and time are infinitely divisible continua, then you reach absurdities. On the other hand, if you assume that space and time are composed of discrete points or moments, then you also reach absurdities. Zeno's aim was to show that the notion of things having parts was incoherent and that, as Parmenides said, all is one. For our purposes though, the relevance of his paradoxes lies in the problems they reveal in the notion that a continuum is composed of discrete points. This problem is critical to the arithmetization of geometry.

2.4 Socrates and Plato: Definition & Categories

2.4.1 Background

Now we leave the mathematical path and consider Socrates' and Plato's development of Pythagorean mathematical and physical ideas into a theory of knowledge. They were so successful that they defined the theoretical framework for nearly all subsequent Western-philosophical debate about knowledge. In the epistemology of Socrates and Plato, word magic reaches a new level of sophistication.

There is considerable doubt as to whether the ideas presented in Plato's dialogues are Socrates' own or Plato's. For our purposes, it doesn't much matter, since we will treat them as a unit. Plato (c. 428–347 BCE), who was the most important student of Socrates (c. 470–399 BCE), is one of the key figures in the history of philosophy. It has been truly said that Western philosophy is merely footnotes to Plato (Kaufmann, *PC*, Vol. I, p. 98). According to Burnet (*GPI*, Ch. IX), it is very likely that Socrates was a Pythagorean; Aristotle also thought his ideas were Pythagorean (*Met.*

987a–b). You may decide for yourself as we investigate his views.[20]

2.4.2   Method of Definition

We have seen the importance to the Pythagoreans of *logoi*: ratios, terms, words, and rational accounts. Therefore, Socrates' probable Pythagoreanism will explain the importance he attached to words. In fact, a shift of emphasis from facts to words was the essence of his contribution to philosophy:

> We know from Plato that the new method of Sokrates consisted precisely in the consideration of things from the point of view of propositions ($\lambda \acute{o} \gamma o \iota$) rather than from that of facts ($\acute{\epsilon} \rho \gamma \alpha$) ...
> (Burnet, *GPI*, p. 146)

An important example of this is his Method of Definition, which is based on the belief that we do not understand something unless we can define it, and that therefore definition should be the principal activity of philosophers. The idea is essentially Pythagorean: to define means to make something definite, and to make it definite is to bound it and set it off from other things. Recall the Pythagoreans' concern with the (de)finite and the in(de)finite. For the ancient Greeks, to be intelligible was to be definite (and hence defined).[21] You can see why definitions would be so important to a Pythagorean like Socrates. Thus, many of the dialogues have as their goal the definition of such terms as *excellence*, *courage*, and *piety*:

> ... what is that common quality, which is the same in all these cases, and which is called courage? (*Laches* 191e)

> Well then, show me what, precisely, this ideal is, so that, with my eye on it, and using it as a standard, I can say that any action done by you or anybody else is holy if it resembles this ideal, or, if it does not, can deny that it is holy. (*Euthyphro* 6e)

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[20]Needless to say, there is an enormous literature on Socrates. I. F. Stone's 1988 book provides a nice overview of his philosophical ideas and how they led to his execution. This is perhaps not a majority opinion among scholars (Griswold, *SGP*), but I find it convincing. More traditional views are presented in Brickhouse & Smith (*SoT*). Burnet (*GPI*, Chs. 8–10) has an interesting account of the historical Socrates, which emphasizes the Pythagorean connections, although, again, this position is considered extreme by many scholars.

[21]See also Section 2.2.3 and p. 45.

> And so of the excellences, however many and different they may
> be, they all have a common nature which makes them excellences.
> (*Meno* 72)

The emphasis on definition continues in philosophy to the present day. Most
knowledge representation schemes in AI and cognitive science are likewise
based on formal structures that represent a concept in terms of its defining
properties ("that common quality" or "common nature").

2.4.3 Knowledge vs. Right Opinion

Socrates' entire theory of knowledge is centered on words, for he claimed
that we truly know something only when we can give a verbal account of
it. The Pythagorean orientation is apparent: something is rational or logical
only when it can be expressed in terms of ratios and *logoi* (words, propo-
sitions, verbal accounts). As he says in the *Laches* (190c), "that which we
know we must surely be able to tell." (See also *Meno* 96d–100a.) Of course
Socrates recognized that many people are skillful in their endeavors, and yet
unable to explain what they're doing in theoretical terms. Yet he denigrated
this atheoretical, practical knowledge, and called it (merely) "right opinion."
Such people, he said, knew what to do, but not why they should do it. He
contrasted this with theoretical knowledge, which for him was the only true
knowledge:

> it is not an art[22] [*techne*] but a practice [*empeiria*], because it can
> produce no principle in virtue of which it offers what it does, nor
> explain the nature thereof, and consequently is unable to point
> to the cause of each thing it offers. And I refuse the name of art
> to anything irrational.[23] (*Gorgias* 465a)

An art, as opposed to a practice, "has investigated the nature of the subject
it treats and the cause of its actions and can give a rational account of each
of them" (*Gorgias* 501a). For a concrete example, consider tuning a lyre. A
musician can do it, but doesn't know why his technique works. He doesn't
have true knowledge. Pythagoras, on the other hand, can give a rational
account (in all senses of rational).

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[22]'Art' (*technē*) must be taken here to mean a systematic or methodical craft, or even
an applied science; on the other hand, a 'practice' (*empeiria*) is based on experience or
practice (*LSJ*, s.vv. τέχνη, ἐμπειρία; Peters, *GPT*, s.v. '*technē*').

[23]N.B. our discussion of *irrational*, p. 19.

2.4.4   The Platonic Forms

The Socratic/Platonic theory of "forms" has been one of the most influential epistemological theories in Western philosophy.[24] It is most comprehensible when seen as an outgrowth of Pythagorean mathematics.

In ancient times — as now — it was held that the truths of mathematics are the most certain truths of all. Two plus two is exactly four; it's not possible that refined measurements will show it's 4.00001, and it's not possible that new discoveries will require this law to be rejected. Other examples of mathematical truths are the Pythagorean theorem, and the theorem that the angles of a triangle add to two right angles. But even if we grant the certainty of these truths — that they are *necessary* truths — we may still question what they are about. They're about numbers or triangles you say? But what is a triangle? Surely not the triangle we draw, which can never have perfectly straight edges, or be made of edges with no width. But these are the only triangles that exist, in the sense that physical objects exist. We may say that mathematical truths are about "idealized" triangles, which are products of thought. But it's clear that the truths of mathematics are objective; all rational investigators will find the angles of a triangle to be two right angles. Hence the triangles of mathematics must have an existence that is not physical, and yet is independent of individual mathematicians. Thus it seems that the only explanation for the objectivity of mathematics is that there is a "realm" where there exist the true, perfect, ideal lines, points, triangles, and other objects of mathematics. The mathematician explores this realm by a process of pure reason.[25]

But if mathematicians are exploring the realm of ideal mathematical objects by pure reason, then why do they draw the figures and constructions that are so prominent in mathematical proofs? Plato's answer (*Republic*

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[24]The theory of forms is discussed in many of the Platonic dialogues. The following are a few key sources: Approximations to an ideal: *Phaedo* 74a–75d; How being and becoming are known: 78d–80b; Philosophy frees us from senses: 82d–83e; How the forms are related to sensible objects: 99d–102d; A summary: *Parmenides* 128e–130a; Critique of the theory: *Parmenides* 130a–135c; Investigation of nature of knowledge: *Theaetetus.* Plato explains the realms of being and becoming in three famous "myths": *Republic* 506d–511e, 514a–519b.

[25]We are running roughshod over many important issues in the philosophy of mathematics, only a few of which will be treated later. The nature of mathematical objects and mathematical truth are still controversial topics. A good reference is (Benacerraf & Putnam, *PM*).

510d–e) was that these are merely aids to the intuition. True intelligence passes beyond the need for these crutches and can proceed by reason alone.

The example of mathematics is easily extended. The physical triangles in the everyday world of sensation are approximations to the ideal triangles that the mathematician studies. Similarly, when we say that two objects are equal, we recognize that this equality is an approximation to mathematical (perfect) Equality. Furthermore, Plato claimed that the triangles and equalities of sense can be understood only by reference to the corresponding ideals. Now, since the Pythagoreans already believed that everything is number (p. 29), it's not such a big step to see the transient and imperfect virtues of individual people or things as approximations to an eternal, idealized and perfect Virtue that exists in the same realm as the mathematicians' triangles. Also, it's not such a big leap to say that these individual virtues can be understood only by reference to ideal Virtue. Philosophers, like mathematicians, are after eternal certainties, and so they investigate the ideals by pure reason. The objects of sense may prod the intuition, but ultimately they mislead.

One effect of this view has been the prevalence in early Western philosophy of *rationalism,* the view that pure reason is a much surer way to the truth than empirical investigation. 'Rationalism' is not a synonym for 'reasonableness'; rather it is a technical term referring to

> (a) the belief that it is possible to obtain by reason alone a knowledge of what exists; (b) the view that knowledge forms a single system, which (c) is deductive in character; and (d) the belief that everything is explicable, that is, that everything can in principle be brought under the single system. (Flew, *DP*, s.v. 'rationalism')

In this sense, rationalism is not the same as the practice of being rational, in the sense of being reasonable. Indeed, a significant question is whether rationalism *is* reasonable. Rejection of rationalism was a major feature of the scientific revolution in the sixteenth century. In a broader sense, Plato's views lead to *intellectualism,* the view that theoretical knowledge is the only true knowledge, and that so-called practical knowledge is "mere opinion" (Section 2.4.3). Intellectualism was not questioned by the scientific revolution, and it is a major background assumption of traditional AI and cognitive science, which tend to focus on intellectual and verbal skills to the exclusion of manual and other nonverbal skills.

The Pythagorean influence is very apparent in the Platonic distinction between Being and Becoming. Recall that the Pythagoreans consigned motion and change to the Indefinite (p. 27). Change was intelligible only when it could be reduced to ratios. Zeno's paradoxes of motion only reinforced this assessment (Section 2.3.3). Yet in the everyday world of sense, things are always changing; everything is in a state of becoming. Thus, the world of sense is in a fundamental way unintelligible, and can be understood only to the extent that it approximates the eternal (changeless) ideals in the world of Being. We can never have scientific knowledge about becoming; knowledge is always of being (p. 41).

The notion of approximations to an ideal is connected with the distinction between being and becoming. The approximations are "striving" or "tending" to *become* the goal, but they will never *be* it. This is illustrated in Zeno's paradoxes of motion (Section 2.3.3). As Burnet (*GPI*, p. 156) says, "The problem of an indefinite approximation which never reaches its goal was that of the age." But a theory of limits did not come for two millennia.

The foregoing ideas are brought together in the theory of forms, but before I discuss it it's necessary to discuss terminology. The Greek words here translated *form* are $\hat{\epsilon i}\delta o\varsigma$ (*eidos*) and $\iota\delta\acute{\epsilon}\alpha$ (*idea*; the source of English *ideal*).[26] These words are often translated *idea* (and thus one hears of Plato's Theory of Ideas), but that is a poor translation, since Plato's "ideas" are definitely not in the head. These words originally meant the form of a thing, its shape, or figure. It is significant that these words were also used to refer to the Pythagorean figures. This is evidence for the view that the theory of forms is a development of Pythagoreanism. Later these words came to mean a characteristic property or category. Notice the continuing assumption that categories are *formal*. (Recall also the discussion on p. 22.)

In Plato's theory of forms two realms are postulated: the familiar realm of sensible objects and the realm of the forms. The realm of sense is characterized by flux and approximation. It is intelligible only to the extent that the sensible objects approximate the ideal forms. The forms themselves are changeless, ideal and perfect. Perception is a faulty source of knowledge; it informs us of the world of sense, which is unintelligible, and can at best hint at the forms. Knowing the forms requires pure reason. Reason is capable of comprehending the forms because the categories of thought are in fact the

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[26]See Peters (*GPT*, pp. 46–47), Taylor (*VS*, Ch. 5), *LSJ* (s.v. $\hat{\epsilon i}\delta o\varsigma$, $\iota\delta\acute{\epsilon}\alpha$), Donnegan (*Lex.*, s.v. $\hat{\epsilon i}\delta o\varsigma$, $\iota\delta\acute{\epsilon}\alpha$), and Burnet (*GPI*, pp. 49–53).

forms. The words we use for these categories (triangle, equality, virtue, etc.) are the names of the forms. True knowledge is thus knowledge of the forms and their logical relations.

Since the forms correspond to what are commonly called categories and concepts, we can draw from the theory of forms the following conclusions about categories and conceptual knowledge:

Categories are real   because they exist in the world of forms. Therefore there is nothing arbitrary or subjective about them.

Categories are static   since there is no change (becoming) in the world of being. Therefore, categories do not evolve.

Categories are a priori   because they exist independent of experience; they are not derived from experience.

Categories are context-independent   because they fit into an eternal logical structure.

Categories are discrete   because they correspond to *terms* (words), and terms are discrete.

Words have definite meanings   because each word names a form, which is definite.

Categories have an objective logical structure.   That is, the logical relations between categories are like those between the mathematical objects.

Terms have objectively correct definitions   because the definitions are determined by the logical structures of the forms that the terms name.

All knowledge is formal knowledge   because the only true knowledge is knowledge of the forms and their logical structures.

What we truly know we can say   because words correspond to forms, and the logical structure of language — properly used — reflects the logical structure of the forms.

These assertions have been assumed — almost without question — throughout most of Western intellectual history, but especially in epistemology, cognitive science and artificial intelligence. Thus it is especially significant that they are rejected by *connectionism*, the new theory of knowledge which is the subject of the second half of this book.

2.4.5   Summary: Socrates and Plato

We have presented — very briefly — what is probably the most influential theory of knowledge and concepts in Western philosophy. In effect it provides a justification for the Pythagorean program. If the only true knowledge is knowledge of the forms, and if the forms are real discrete objects fitting into a logical structure, then such knowledge can be expressed verbally, as terms arranged in formal structures. Thus the truly real world, the world of Being, has a rational structure, even if the sensible world, the world of Becoming, which is only a distorted shadow of true reality, is ultimately irrational and unintelligible. The principal task of philosophy and science thus becomes the charting of the formal structure of the world of forms.

2.5   Aristotle: Formal Logic

2.5.1   Background

Aristotle (384–322 BCE) is one of the most influential thinkers in Western philosophy. If he stands behind Plato it is only because he was a student of Plato, and thus is the principal "footnoter" of his teacher (p. 39).[27]

Aristotle's scholarship had enormous breadth: he wrote on nearly every subject from logic, physics and biology, to love, music and table manners. He was nothing if not prolific: one ancient catalog[28] lists 150 books (about 50 modern volumes) comprising 445 250 lines! And this catalog is known

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[27]Two readable summaries of Aristotle's philosophy are Randall (*Aris.*) and Barnes (*Aris.*). There are many books of selections from Aristotle's works.

[28]Diogenes Laertius, 5.22–27

to be incomplete! (Barnes, *Aris.*, p. 3) Unfortunately, only about a fifth of Aristotle's writings have survived the accidents of time and the hands of the book burners. It is no wonder that throughout most of history Aristotle has been known simply as "The Philosopher." Here we will be concerned only with Aristotle's logical works; these are the ones that have been most influential in the traditional theory of knowledge.

2.5.2 Structure of Theoretical Knowledge

Recall Socrates' distinction between knowledge and right opinion (p. 41). Knowledge is preferable because it's more reliable. That is, if we just have right opinion, then we only know that what we are doing has worked in the past; we cannot be certain that it will work the next time we try it. On the other hand, if we have knowledge, then we can give a rational account of what we do. Therefore, since we know the necessary connections between things we do not have to fear being wrong.

Aristotle accepts this same basic definition, since he too expects true knowledge to be universal.[29] Also, like Plato, he sees that the only way to achieve this universality is to give knowledge a strict deductive (logical) structure grounded in indubitable premisses.[30] This leads to two subgoals in Aristotle's investigation of the structure of knowledge: one is to set down the rules for deductive argument; the other is to determine how we can know the primary truths, since these cannot be established deductively. We will discuss the results of Aristotle's investigations in each of these areas.

2.5.3 Primary Truths

The primary truths must be more than mere assumptions, since in that case the conclusions drawn from them would be no better than assumptions. Further, the primary truths cannot be merely hypotheses, since then the

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[29]Aristotle distinguishes three kinds of scientific knowledge (*Metaphysics* 6.1.1025b25): theoretical knowledge, productive knowledge and practical knowledge. At the present time we are concerned only with theoretical knowledge, and when we use the term *knowledge* this is what it will mean.

[30]Aristotle, like Plato, was an epistemological "realist," which means that he took the forms to exist independently of us and not to be creations of our minds. He differed from Plato in putting the forms in the objects of sensation rather in a separate ideal realm. However, these ontological distinctions are not relevant to our concerns here.

conclusions would be no surer than hypothetical. For scientific knowledge to be absolutely certain, the primary truths themselves are required to be absolutely certain. But since the primary truths are the starting point of deductions they cannot themselves be established deductively. Therefore the primary truths must be *self-evident*, in the literal sense of providing their own evidence. That is, the primary truths are self-justifying.

In most cases Aristotle takes the primary truths to be definitions or parts of definitions. But again we must be careful, since Aristotle understands definition differently from the way we do now. In the deductive sciences we usually take a definition to be a *prescription* for the use of a word. That is, a definition is way of introducing a word as an abbreviation for a longer sequence of words. Such definitions are not truths, but conventions, and therefore would lead to no more certainty than arbitrary assumptions.

The modern notion of definition will not serve Aristotle's needs. For him a definition is a factual statement that says what it is for a thing to be what it is. An example will make this notion clearer.

We can begin with a good example of a definition: 'a triangle is a three-sided figure'.[31] The purpose of this definition is *not* to introduce 'triangle' as an abbreviation for 'three-sided figure', nor is it even to explain the way the word 'triangle' is used in English. Rather, its purpose is to state what it is for something to be a triangle. As it's usually put, the definition states the *essence* of triangles: the properties that anything must have in order to be a triangle. Something that's not a figure, or that's not three-sided, is surely not a triangle. Conversely, any figure with three sides is surely a triangle.

Traditionally, definition in terms of essences is considered the hallmark of Aristotle's theory of definition, and much medieval (and even modern) philosophy was concerned with the nature of essences. Yet it's remarkable that there is not a Greek word corresponding to the translation *essence*. The phrase most commonly translated *essentially* is καθ' αὑτό (*kath' hauto*), which means *per se*, or *in itself*. So where we often read "what is Man essentially?" or "what is Man in essence?", we should read "what is Man in itself?" Similarly, there is no single word corresponding to *essence*. The phrase most commonly translated this way is τὸ τί ἐστι (*to ti esti*), which means *the 'What is it?'* (a question turned into a noun). Another such phrase

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[31]The definition we use for this example, 'a three-sided figure', admits triangles bounded by curved lines, and these are traditionally called triangles. However, for the purpose of the example we restrict our attention to *rectilinear* figures and triangles, that is, those bounded by straight lines.

is τὸ τί ἦν εἶναι (*to ti ēn einai*), which means something like *the 'What is it to be what it is?'*. These translations are more awkward, but more accurate (Randall, *Aris.*, p. 47, n. 13). We will avoid *essence* and derivative terms.

The problem of *essence* is a good illustration of the role of language in the history of ideas. The Latin *essentia* was coined, perhaps by Cicero, to translate the Greek *ousia* (one's own, property, being); in the Medieval period it came to mean *essence* in the sense under consideration here.[32] Over the two millenia since its invention, much ink has been spilled about essences — what they are, where they are, and so forth.

But observe: by creating a word, Cicero (or whoever) created a philosophical problem. Once the word *essentia* had been invented and used in meaningful contexts, such as translations and paraphrases of Aristotle, it was necessary to find something that it named. The implicit presumption is that if it can be used meaningfully, then it must mean *something*, in other words, there must be *some things* (essences) to which the word refers.

The existence of a word such as *essence* can also bias the way we go about our investigations and can set bounds on acceptable answers. If we begin our inquiry by seeking "the essence of life," we will likely find the soul or an *élan vital* or some such. On the other hand, if we begin by asking, "What is it to be alive?" then we are more likely to come up with a description of a process, or at least an operational test for life. Thus we must beware of the "bewitchery of words."

Back to Aristotelian definitions. Since they state the most fundamental properties of things, their discovery may require significant analysis and scientific investigation. Once found, however, they are self-evident in the way illustrated above. Who could rationally deny that a triangle is a three-sided figure?

A proper Aristotelian definition contains one or more primary truths. For example, in the definition of triangle we may see two primary truths: that a triangle is a figure, and that a triangle has three sides. From these and other primary truths many derivative truths follow in turn by deduction. In summary, Aristotelian definitions are *self-evident matters of fact*, not prescriptions.

There is one more aspect of Aristotle's approach that we must address before leaving the topic of primary truths. This is that Aristotle permits the

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[32]At least as early as Thomas Aquinas (1225–1274), e.g., *Sum. Theol.* 1, q.3, a.3 concl. and q.29, a.2 ad 3.

various sciences to have their own primary truths; he does not seek to derive all truths from one first principle, as Plato did. As we'll see, Euclid, following Aristotle, deftly avoids the chasm between arithmetic and geometry — the discrete and the continuous — by basing each science on its own primary truths (p. 53).

Unfortunately, it's much more difficult to apply Aristotle's idea of definition outside of mathematics. What is the definition of cow, or person? To define person we must find those properties without which a thing would not be human.

2.5.4   Formal Logic

Although many earlier philosophers had studied the forms of arguments (especially the Sophists), we owe to Aristotle the founding of logic as a science. He was the first to analyze propositions into *terms* and to show how deductive processes rearrange these terms (recall p. 22). For example, consider the well-known syllogism:[33]

$$
\begin{array}{rl}
 & \text{All men are mortal;} \\
 & \text{Socrates is a man;} \\
\hline
\text{therefore,} & \text{Socrates is mortal.}
\end{array}
$$

The validity of this argument does not depend on the particular terms 'Socrates', 'man' and 'mortal' that appear in it; indeed, they are like game tokens (calculi, p. 24). All that's important to the validity of the argument is its *form* (hence, *formal* logic).

The general form of this argument can be expressed in a formal rule, or *schema*, such as this:

$$
\begin{array}{rl}
 & \text{All } M \text{ is } P \\
 & S \text{ is } M \\
\hline
\text{therefore,} & S \text{ is } P
\end{array}
$$

Indeed, Aristotle was the first to use variables (such as S, M and P here) to express rules formally; it is a major contribution and a model for rule-based systems in AI and cognitive science.

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[33]I retain the conventional translation 'man' for ἄνθρωπος (*anthrōpos*), which, though masculine in gender, was generally used for people of both sexes.

Aristotle considered all the possible arrangements of the terms in syllogisms and classified them into three *figures* (*schemata*, p. 22). The preceding example is in the first figure; here is a valid syllogism in the second (Joseph, *IL*, p. 258):

<div align="center">

No insects have eight legs;
Spiders have eight legs;
_____
therefore,   Spiders are not insects

</div>

In general:

<div align="center">

No P is M
S is M
_____
therefore,   No S is P

</div>

The three figures enumerate the possible arrangements of the three terms that occur in the syllogism: S the subject of the conclusion, P the predicate of the conclusion, and M the *middle term*, which appears in both premisses but not in the conclusion. Writing the terms of the propositions in the order subject-predicate, we have the three figures:

<div align="center">

$$\begin{array}{ccc} MP & PM & MP \\ \underline{SM} & \underline{SM} & \underline{MS} \\ \\ SP & SP & SP \end{array}$$

</div>

Note that this exhausts all possible arrangements, if the order of the premisses is not considered.

Aristotle's formal logic can be considered a continuation of the Pythagorean program. The earliest Pythagoreans thought that things were literally composed of numbers, that is, units (terms) arranged in various forms or figures. Later Pythagoreans believed a more abstract version of this theory: that every thing had a number through which it could be understood. Aristotle moves to a higher level of abstraction, since for him it's not things that are formal arrangements of terms, but knowledge itself. What has not changed is the identification of the intelligible with formal structures.

2.5.5   Epistemological Implications

We now turn to some of the epistemological implications of Aristotle's view. Since for Aristotle definitions are matters of fact, there is one correct definition for each term, that is, the definition is a formula (*logos*) saying "what it is to be what it is." Like Socrates and Plato (Section 2.4.2), Aristotle believed that the meaning of a term can be expressed exactly in a finite formula.

Similarly, as we've seen, Aristotle was able to express his deductive rules formally — as mechanical symbol manipulation processes. Therefore, in Aristotle's ideal of a completed science, all the knowledge is expressed as formal (structural) relationships between symbol structures (schemata, formulas).

We summarize the epistemological implications of Aristotle's theory:

- Definitions are objective matters of fact, which can be expressed in finite formulas.

- Deduction can be described by the formal manipulation of terms arranged in specified schemata.

- A completed science takes the form of propositions connected formally to definitions.

These assertions have become incorporated into our unconscious assumptions about "true knowledge," and they provide the ultimate source of the formal, deductive knowledge representation and inference schemes commonly employed in cognitive science and artificial intelligence. However, in volume 2 we will see that they are assumptions that need to be questioned, and in fact rejected.

2.6   Euclid: Axiomatization of Continuous & Discrete

Euclid alone has looked on Beauty bare.

— Edna St. Vincent Millay (*The Harp Weaver*, 4, sonnet 22)

2.6.1 Background

We return now to the mathematical part of our story, and consider an important investigation of the continuous and discrete in mathematics. Eudoxus (408–355 BCE), a student of Plato, was probably the greatest Greek mathematician before Archimedes.[34] It is likely that he originated both the theory of magnitudes and the method of exhaustion, which we find in Euclid's *Elements*. Yet not one of his works survives (Bochner, *RMRS*, p. 325). On the other hand, by all accounts (ancient and modern) Euclid (c. 300 BCE) was a rather mediocre mathematician. Nevertheless, the 13 books of his *Elements* have survived intact, and have been a required subject in school from his time until well into the twentieth century. Its apparently perfect reduction of a body of knowledge to a deductive structure has an austere beauty, as Millay and many others have recognized.

2.6.2 Axiomatic Structure

Euclid's *Elements* is an application to mathematics of Aristotle's idea of a science as defined in his two major logical works, the *Prior* and *Posterior Analytics* (Maziarz & Greenwood, *GMP*, p. 242–243). It begins with definitions in terms of necessary and sufficient attributes that are taken to be prior to the term defined. It bases its deductions on axioms ("common notions"), which are taken to be self-evident truths, and postulates, which are taken as the starting points of the particular science (*Post. An.* 74b5–77a30). The organization of the whole makes its deductive structure explicit, since no proposition may be admitted unless it is deducible from the first principles.[35] The *Elements* was thus the first concrete demonstration of how a body of knowledge could proceed by formal operations from explicitly given hypotheses. It remained the exemplar of formal reasoning until some of its defects were discovered in the nineteenth century.

The Platonic/Aristotelian view of knowledge as a formal structure of discrete propositions is further evident in the use of the term *elements*. Pre-

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[34]A general source for the material in this section is Maziarz & Greenwood (*GMP*, Part 4). The definitive translation of Euclid's *Elements* is Heath (*Euclid*).

[35]Aristotle's use of rational necessity should be contrasted with Plato's. Plato used rational analysis as a means of regression from the familiar forms back to the most basic form. Aristotle takes certain principles as given, and the then by rational synthesis shows how various conclusions follow from them by rational necessity (Maziarz & Greenwood, *GMP*, p. 242–243).

Euclidean mathematicians had already organized theorems by showing that many of them followed from a few general principles, which they called *elements*, by analogy with the alphabet's relation to language (Maziarz & Greenwood, *GMP*, p. 240). Compare Plato's notion of the unanalyzable *elements* of which the "syllables" of knowledge are composed (*Theaetetus* 201d–206b). In both cases there is a presumption that knowledge is a complex of discrete, indivisible elements. This view is characteristic of the traditional view of knowledge, as we will see (Sections 3.6.1).

2.6.3   Theory of Magnitudes

The Pythagoreans and Zeno had demonstrated the difficulty of having a single theory that encompasses both discrete and continuous quantities. Therefore, Euclid axiomatized each of them separately. In Book 7 he develops the theory of discrete quantities — what we call number theory. However, in Book 5 he develops an axiomatic theory of continuous quantities, or *magnitudes*. This is based on relations of proportion, that is, on *ratios*. Using this theory he is able to prove the very important *principle of continuity*, which is the basis for the *method of exhaustion* — a way of finding the limits of sequences (p. 55). This principle shows that certain infinite series must eventually get smaller than any number we can pick. We consider briefly Euclid's theory.

Just as numbers (i.e. integers) are idealizations of discrete objects, such as pebbles or tokens, taken as members of ensembles, so magnitudes are idealizations of continuous quantities, such as lengths and areas. Both idealizations are based on intuitions about the familiar world. For example, we see we are surrounded by discrete objects. We also see continuous change, such as continuous motion, growth, and the flow of time.

Although we have basic intuitions of both the continuous and the discrete, our Pythagorean view of knowledge has caused us to view numbers as more basic than magnitudes — hence the goal of *arithmetizing geometry* (Chapter 4). We will see in volume 2 that we can as easily *geometrize arithmetic*, that is, reduce the discrete to the continuous.

Euclid's theory of magnitudes is not expressed with nearly as much rigor as would be demanded now. In contrast to the axiomatization of geometry in Book 1, where point, lines, and the like are defined, the basic concept *magnitude* is not defined at all. The definitions we find in Book 5 have to do with multiples, ratios, proportions, and so forth. Further, there are
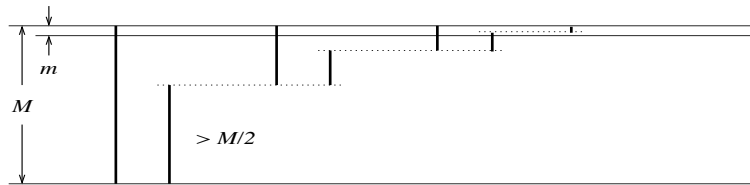
Figure 2.16: Principle of Continuity. M and m are two unequal magnitudes, $M > m$. Subtract from M a magnitude $> M/2$ and consider the remainder. Subtract at least a half of the remainder, and continue. Eventually a magnitude smaller than m will remain.

no postulates for magnitudes. Rather, the proofs are based on the axioms (Common Notions) from Book 1 together with informal intuitions about magnitudes.[36]

The Principle of Continuity, which is the basis for Euclid's method of handling limits, is of fundamental importance for the eventual arithmetization of geometry:

> Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out ... And the theorem can be similarly proved even if the parts subtracted be halves. (Euclid, Bk. 10, Prop. 1)

See Fig. 2.16, in which M is the larger magnitude and m the smaller. The Principle of Continuity is used in method of exhaustion, discussed next.

The method of exhaustion circumvents the difficulties with infinitesmals and infinite processes pointed out by Zeno (Section 2.3). It accomplishes this by replacing actual infinities by potential infinities. For example, Euclid wants to prove that the areas of circles are to one another as the squares of their diameters. He has already proved this theorem for regular polygons, so he would like to make use of Antiphon's insight that a circle can be thought of as a circle with an infinite number of sides (Fig. 2.17). In modern terms,

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[36]The Common Notions are: (1) Things which are equal to the same thing are also equal to one another. (2) If equals be added to equals, the wholes are equal. (3) If equals be subtracted from equals, the remainders are equal. (4) Things which coincide with one another are equal to one another. (5) The whole is greater than the part. (Euclid, Bk. 1)
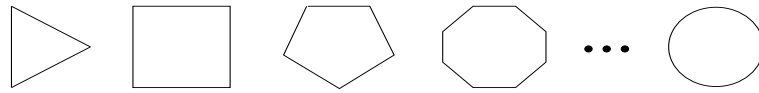
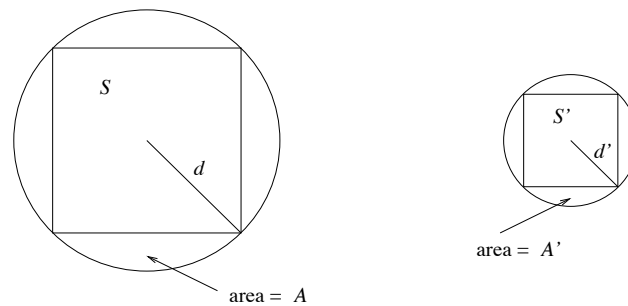Figure 2.17: The Circle as an Infinite-Sided Polygon

Figure 2.18: Example Application of Method of Exhaustion

he would like to "take the limit" and let the number of the polygon's sides go to infinity. But, instead of depending on the problematic notion of an infinite-sided polygon, Euclid applies the principle of continuity, and shows that the difference between the circle and polygon can be made smaller than any given magnitude by increasing the number of sides sufficiently. This allows him to show that a contradiction would result from the assumption that the area is different from that given by the ratio of the squares of the diameters.

The method is basically this (Fig. 2.18). Let A be the area of the larger circle. Contrary to the theorem, assume the area given by the ratio of the squares is $B < A$, that is, $A'(dd/d'd') = B < A$. Inscribe a polygon with sufficient sides so that its area is $S > B$. A similar polygon S' is constructed in the smaller circle. But since the areas of the polygons are as the squares of the diameters, it can be shown that the area of the larger polygon is less than B. Specifically, $S/S' = dd/d'd' = B/A'$. Hence $S/B = S'/A'$. But $S' < A'$, so $S < B$, which contradicts the fact that it was constructed with area greater than B. A contradiction similarly follows from the assumption $B > A$. Hence $B = A$.

2.6.4 Summary

Euclid made the deductive structure of mathematics explicit through the methods of Aristotle. However, the inability of Greek mathematics to reconcile the rational and irrational forced him to treat continuous and discrete quantities separately. In particular, he was not able to rationalize the continuous by arithmetizing geometry. The continuous and discrete remained unreconciled for over 2000 years. The arithmetization of geometry was finally accomplished around the turn of the twentieth century (see Chapter 4).