Molecular Combinator Reference Manual

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Abstract

This report contains, in summary form, definitions, schematic reactions, and equivalences of all combinators in use by this project. It will be updated as new combinators, equivalences, etc. are required.

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Introduction

1. Most of the combinator definitions and equivalences (beyond those peculiar to molecular computation, such as R, D, and V) are from Curry and Feys [1].

2. We follow the usual convention in combinatory logic of omitting parentheses that associate to the left. For example, XYZ means $((XY)Z)$, and $\mathsf{B(BW(BC))(BB(BB))}$ means $\mathsf{((B((BW)(BC)))((BB)(BB)))}$.

3. In the definitions of the operators, variables are marked with primes (e.g., X') and parenthesized superscripts (e.g., $X^{(4)}$) to indicate shared complexes. See the description of the V (Sharing) Primitive (Section 17).

4. Notice that the following are distinct and have different meanings: X^n (powers of combinators), X_n (polyadic combinators), $X^{(n)}$ (sharing), $X_{(n)}$ (deferred combinators), $X^{[n]}$ (left reduction), $X_{[n]}$ (polyadic extension); see **Other Notation** (p. 17). X_n is also used in the usual way to denote an element in a series $X_1, X_2, X_3, \ldots$. When subscripts and superscripts of any kind are combined, the subscripts take precedence; thus Φ_n^m means $(\Phi_n)^m$.

5. The size $|X|$ of a nonprimitive combinator X is expressed in terms of the number of S, K, and A nodes that it contains. Since nonprimitive combinator definitions are binary trees, if they contain no other nodes besides S, K, and A, then the counts satisfy $A = S + K - 1$, and the total nodes are $T = 2A + 1 = 2(S + K) - 1$.

6. A combinator is called *regular* if it does not affect its first argument, thus,

$$FXY_1 \cdots Y_n \Longrightarrow XZ_1 \cdots Z_m. \tag{1}$$

   Most combinators (e.g., B, $\mathsf{B'}$, C, I, K, S, W, Y, Φ, Φ_n, Ψ) are regular.

7. A combinator is said to be of *order* n if it expects n arguments. Thus I is of order 1, K is of order 2, and S is order 3.

8. If F is a regular combinator, as in Eq. 1, then it is said to be of *degree* m; that is, it produces m combinators beyond the one required for its regularity. For example, I and K are of degree 0, and C and S are of order 2. S_n is of order $n + 2$ and of degree $n + 1$.

Definitions of Combinators

1 A Primitive (Application Complex)

The application (A) complex represents the application of a combinator to its argument. The application of F to X, written FX, is represented by a molecular complex $U\mathsf{A}FX$, in which the "operator" binding site of A is linked to F, the "operand" binding site is linked to X, and the "result" site is linked to U, the complex into which the result of FX will be linked.

All (or most) of the non-terminal (interior) nodes of a combinator tree are A nodes; the terminals (leaves) are primitive combinators (e.g., S and K). If the network is not a tree, but has shared nodes or cycles, then (most of) the non-terminal nodes are A and V (sharing) nodes. (We say "most" because later we may want to define additional interior node types.)

2 B Combinator (Elementary Compositor)

Definition:

$$\mathsf{B}XYZ \Longrightarrow X(YZ) \tag{2}$$

Reduction to SK:

$$\mathsf{B} = \mathsf{S(KS)K} \tag{3}$$

Size: $2\mathsf{S} + 2\mathsf{K} + 3\mathsf{A} = 7$ total.
Equivalences:

$$\begin{aligned}
\mathsf{B} &= \mathsf{CB'} && (4)\\
\mathsf{B} &= \mathsf{C(JIC)(JI)} && (5)\\
\mathsf{B}^n FGX_1 \cdots X_n &\Longrightarrow F(GX_1 \cdots X_n) && (6)\\
\mathsf{B}_{(n)} FX_1 \cdots X_n YZ &\Longrightarrow FX_1 \cdots X_n(YZ) && (7)\\
\mathsf{B}_{[n]} F_0 \cdots F_n X &\Longrightarrow F_0(F_1(\cdots(F_n X)\cdots)),\ n \geq 0 && (8)\\
\mathsf{B}^{[n]} FX_1Y_1 \cdots X_nY_n &\Longrightarrow F(X_1Y_1)\cdots(X_nY_n),\ n \geq 0 && (9)
\end{aligned}$$

Notes: Eq. 2 shows that if F and G are two order-1 (monadic) functions, then $\mathsf{B}FG$ is their composition $F \circ G$ (see also Sec. 28).

If F is regular (p. 2), $FXY_1 \cdots Y_n \Longrightarrow XZ_1 \cdots Z_m$, then

$$\mathsf{B}FGXY_1 \cdots Y_n \Longrightarrow GXZ_1 \cdots Z_m.$$

That is, G is applied to the result of applying F to the arguments $XY_1 \cdots Y_n$; F is performed, then G.

3 B' Combinator (Permuting Compositor)

Definition:

$$\mathsf{B}'XYZ \Longrightarrow X(ZY) \tag{10}$$

Reduction to SK:

$$\mathsf{B}' \;=\; \mathsf{CB} \tag{11}$$

Size: $7\mathsf{S} + 6\mathsf{K} + 12\mathsf{A} = 25$ total.
Equivalences:

$$(\mathsf{B}')^n FXG_1 \cdots G_n \;\Longrightarrow\; F(G_n(\cdots(G_1X)\cdots)),\; n \geq 0 \tag{12}$$

$$\mathsf{B}'_{(n)} FX_1 \cdots X_n YZ \;\Longrightarrow\; FX_1 \cdots X_n(ZY) \tag{13}$$

$$\mathsf{B}'_{[n]} FX_1 \cdots X_{n+1} \;\Longrightarrow\; F(X_{n+1} \cdots X_1),\; n \geq 0 \tag{14}$$

$$\mathsf{B}'^{[n]} FX_1Y_1 \cdots X_nY_n \;\Longrightarrow\; F(Y_1X_1)\cdots(Y_nX_n),\; n \geq 0 \tag{15}$$

4 C Combinator (Elementary Permutator)

Definition:

$$\mathsf{C}XYZ \Longrightarrow XZY \tag{16}$$

Reduction to SK:

$$\mathsf{C} \;=\; \mathsf{S(BBS)(KK)} \tag{17}$$

Size: $6\mathsf{S} + 6\mathsf{K} + 11\mathsf{A} = 23$ total.
Equivalences:

$$\mathsf{C}_{(n)} FX_1 \cdots X_n YZ \;\Longrightarrow\; FX_1 \cdots X_n ZY \tag{18}$$

$$\mathsf{C}_{[n]} FX_1 \cdots X_n X_{n+1} \;\Longrightarrow\; FX_{n+1}X_1 \cdots X_n \tag{19}$$

$$\mathsf{C}^{[n]} FX_1X_2 \cdots X_{n+1} \;\Longrightarrow\; FX_2 \cdots X_{n+1}X_1 \tag{20}$$

$$\mathsf{C} \;=\; \mathsf{JC_*(JC_*)(JC_*)} \tag{21}$$

5 C_* Combinator (Pure Permutator)

Definition:

$$\mathsf{C}_*XY \Longrightarrow YX \tag{22}$$

Reduction to SK:

$$\mathsf{C}_* \;=\; \mathsf{CI} \tag{23}$$

$$\mathsf{C}_* \;=\; \mathsf{JII} \tag{24}$$

Size: $6\mathsf{S} + 6\mathsf{K} + 11\mathsf{A} = 23$ total (Def. 23).
Equivalences:

$$\mathsf{C}_*^n X_1 \cdots X_{n+1} \implies X_{n+1} \cdots X_1,\ n \geq 0 \tag{25}$$
$$(\mathsf{C}_*)_{[n]} X_1 \cdots X_{n+1} \implies X_{n+1}(\cdots(X_2 X_1)\cdots),\ n \geq 0 \tag{26}$$

6 D Primitive (Elementary Deleter)

Reaction:

$$\mathsf{D}p + \mathsf{PQ} \longrightarrow \mathsf{P}p + \mathsf{DQ} \tag{27}$$
$$\mathsf{DA}XY + \mathsf{DQ} + \mathsf{PQ} \longrightarrow \mathsf{D}X + \mathsf{D}Y + \mathsf{PAQ}_2 \tag{28}$$
$$\mathsf{D}U\mathsf{R}X + 2\mathsf{PQ} \longrightarrow UX + \mathsf{P}_2\mathsf{RQ} + \mathsf{DQ} \tag{29}$$
$$\mathsf{D}U\mathsf{V}X + \mathsf{PQ} \longrightarrow \mathsf{P}U\mathsf{V}X + \mathsf{DQ} \tag{30}$$
$$\mathsf{DPV}X + \mathsf{PQ} \longrightarrow \mathsf{D}X + \mathsf{P}_2\mathsf{VQ} \tag{31}$$

Notes: In Eq. 27, p represents any primitive combinator (e.g., S or K). Notice that in Eq. 29, a deletion cancels a replication in progress. However, in Eq. 30, a deletion does not affect a shared complex, except to cap the deleted sharing link.
Reaction Specification:

```
d: D, a: A, x, y, d′: D, p: P, q: Q, q′: Q.
d a, a_1 x, a_2 y, d′ q′, p q
=> (DeleteApplication)
   d x, d y, p a, a_1 q, a_2 q′.

d: D, u, r: R, x, p: P, p′: P, q: Q, q′: Q.
d r_1, u r_2, r x, p q, p′ q′
=> (DeleteReplicator1)
   u x, p r_1, p′ r_2, r q, d q′.

d: D, u, r: R, x, p: P, p′: P, q: Q, q′: Q.
d r_2, u r_1, r x, p q, p′ q′
=> (DeleteReplicator2)
   u x, p r_1, p′ r_2, r q, d q′.

d: D, u, v: V, x, p: P, q: Q.
d v_1, u v_2, v x, p q
=> (DeleteSharing1)
   p v_1, u v_2, v x, d q.

d: D, u, v: V, x, p: P, q: Q.
```

```
d v_2, u v_1, v x, p q
=> (DeleteSharing2)
   p v_2, u v_1, v x, d q.

d: D, p': P, v: V, x, p: P, q: Q.
d v_1, p' v_2, v x, p q
=> (DeleteFinalSharing1)
   p v_1, p' v_2, v q, d x.

d: D, p': P, v: V, x, p: P, q: Q.
d v_2, p' v_1, v x, p q
=> (DeleteFinalSharing2)
   p v_2, p' v_1, v q, d x.

d: D, pc: Prim, p: P, q: Q.
d pc, p q
=> (DeletePrimitive)
   p pc, d q.
```

Notes: In the last (`DeletePrimitive`) rule, '`Prim`' stands for any primitive combinator. Therefore, at least at the present time, that rule must be repeated with '`Prim`' replaced by each primitive combinator species in use (e.g., '`K`', '`S`').

7 I Combinator (Elementary Identificator)

Definition:

$$\mathsf{I}X \Longrightarrow X \tag{32}$$

Reduction to SK**:**

$$\mathsf{I} = \mathsf{SK}X \tag{33}$$

Size: $1\mathsf{S} + 2\mathsf{K} + 2\mathsf{A} = 5$ total (taking $\mathsf{I} = \mathsf{SKK}$).
Equivalences:

$$\mathsf{I} = \mathsf{CK}X \tag{34}$$

$$\mathsf{I} = \mathsf{WK} \tag{35}$$

$$\mathsf{I}_{(n)}X_0 \cdots X_n \Longrightarrow X_0 \cdots X_n \tag{36}$$

8 J Combinator

Definition:

$$\mathsf{J}UXYZ \Longrightarrow UX(UZY) \tag{37}$$

9 K Combinator (Elementary Cancellator)

Definition:

$$\mathsf{K}XY \Longrightarrow X \tag{38}$$

Reaction:

$$U\mathsf{A}_2\mathsf{K}XY + \mathsf{DQ} \longrightarrow UX + \mathsf{DA}_2\mathsf{KQ}Y \tag{39}$$

Equivalences:

$$\mathsf{K}^n XY_1 \cdots Y_n \;\Longrightarrow\; X \tag{40}$$

$$\mathsf{K}_{(n)} X_0 \cdots X_n Y \;\Longrightarrow\; X_0 \cdots X_n \tag{41}$$

$$\mathsf{K}^{[n]} X_1 \cdots X_{2n} \;\Longrightarrow\; X_1 X_3 \cdots X_{2n-3} X_{2n-1},\ n \geq 1 \tag{42}$$

$$(\mathsf{CK})_{[n]} X_1 \cdots X_n Y \;\Longrightarrow\; Y,\ n \geq 0 \tag{43}$$

$$(\mathsf{CK})_{(n)} X_1 \cdots X_n YZ \;\Longrightarrow\; X_1 \cdots X_n Z \tag{44}$$

Reaction Specification:

```
a: A, b: A, k: K, d: D, q: Q, u, x, y.
u a, a_1 b, b_1 k, b_2 x, a_2 y, d q
=> (Kreaction)
   u x, d a, a_1 b, b_1 k, b_2 q, a_2 y.
```

10 N Combinator (Inert Complex)

The N (inert) combinator is used when we want to prevent reduction, generally when we are intending to produce a static structure. For example, if the structure $FX_1 \cdots X_n$ is generated, then there is a risk that the reduction rules for F will destroy the structure. This is avoided by using the inert combinator, e.g. $\mathsf{N}X_1 \cdots X_n$. Since it is inert, there are no reduction or reaction rules for it. Of course, in practice, there need not be just one inert combinator, and any molecular species that does enter into the computational reactions could be used.

11 P Primitive (Result Cap)

The result cap is inert; it is a place-holder for the "result" binding-site of any group.

12 Q Primitive (Argument Cap)

The argument cap is inert; it is a place-holder for the "argument" binding-site of any group (in particular, for the "operator" and "operand" sites of an A complex).

13 R Primitive (Elementary Replicator)

Reaction:

$$UV\mathsf{R}p + \mathsf{P}p + \mathsf{PQ} \longrightarrow Up + Vp + \mathsf{P}_2\mathsf{RQ} \tag{45}$$

$$UV\mathsf{RA}XY + \mathsf{PAQ}_2 + \mathsf{P}_2\mathsf{RQ} \longrightarrow UV\mathsf{A}_2\mathsf{R}_2XY + 3\mathsf{PQ} \tag{46}$$

Notes: In Eq. 45, p represents any primitive combinator (e.g., S or K).
Reaction Specification:

```
r: R, a: A, u, v, x, y, r′: R, a′: A,
   p: P, p′: P, p′′: P, q: Q, q′: Q, q′′: Q.
u r_1, v r_2, r a, a_1 x, a_2 y,
   p r′_1, p′ r′_2, r′ q, p′′ a′, a′_1 q′, a′_2 q′′
=> (ReplicateApplication)
   u a, v a′,
   a_1 r_1, a′_1 r_2,
   a_2 r′_1, a′_2 r′_2,
   r x, r′ y,
   p q, p′ q′, p′′ q′′.

r: R, pc: Prim, u, v, pc′: Prim, p: P, p′: P, q: Q.
u r_1, v r_2, r pc, p q, p′ pc′
=> (ReplicatePrimitive)
   u pc, v pc′,
   p r_1, p′ r_2, r q.
```

Notes: In the last (`ReplicatePrimitive`) rule, ‘`Prim`’ stands for any primitive combinator. Therefore, at least at the present time, that rule must be repeated with ‘`Prim`’ replaced by each primitive combinator species in use (e.g., ‘`K`’, ‘`S`’).

14 S Combinator (Elementary Distributor, Replicating)

Definition:

$$\mathsf{S}XYZ \Longrightarrow XZ(YZ) \tag{47}$$

Reaction:

$$U\mathsf{A}_3\mathsf{S}XYZ + \mathsf{P}_2\mathsf{RQ} \longrightarrow U\mathsf{A}(\mathsf{A}X)(\mathsf{A}Y)\mathsf{R}Z + \mathsf{PS} + \mathsf{PQ} \tag{48}$$

Reaction Specification:

```
a: A, a′: A, a′′: A, s: S, r: R, p: P, p′: P, q: Q,
   u, x, y, z.
```

```
u a, a_1 a', a'_1 a'', a''_1 s, a''_2 x, a'_2 y, a_2 z,
   p r_1, p' r_2, r q
=> (Sreaction)
   u a, a_1 a', a'_1 x, a'_2 r_1,
        a_2 a'', a''_1 y, a''_2 r_2,
        r z,
   p s, p' q.
```

Equivalences:

$$\mathsf{S} = \mathsf{B}(\mathsf{B}(\mathsf{BW})\mathsf{C})(\mathsf{BB}) \tag{49}$$

$$\mathsf{S}^{[n]} X Y Z_1 \cdots Z_n \Longrightarrow X Z_1 \cdots Z_n (Y Z_1 \cdots Z_n) \tag{50}$$

15 $\check{\mathsf{S}}$ Combinator (Elementary Distributor, Sharing)

Definition:

$$\check{\mathsf{S}} X Y Z \Longrightarrow X Z' (Y Z) \tag{51}$$

Reaction:

$$U \mathsf{A}_3 \check{\mathsf{S}} X Y Z + \mathsf{P}_2 \mathsf{V} \mathsf{Q} \longrightarrow U \mathsf{A}(\mathsf{A} X)(\mathsf{A} Y) \mathsf{V} Z + \mathsf{PS} + \mathsf{PQ} \tag{52}$$

Reaction Specification:

```
a: A, a': A, a'': A, s: Ssh, v: V, p: P, p': P, q: Q,
   u, x, y, z.
u a, a_1 a', a'_1 a'', a''_1 s, a''_2 x, a'_2 y, a_2 z,
   p v_1, p' v_2, v q
=> (SharingSreaction)
   u a, a_1 a', a'_1 x, a'_2 v_1,
        a_2 a'', a''_1 y, a''_2 v_2,
        v z,
   p s, p' q.
```

Equivalences:

$$\check{\mathsf{S}} = \mathsf{B}(\mathsf{B}(\mathsf{B}\check{\mathsf{W}})\mathsf{C})(\mathsf{BB}) \tag{53}$$

$$\check{\mathsf{S}}^{[n]} X Y Z_1 \cdots Z_n \Longrightarrow X Z_1' \cdots Z_n' (Y Z_1 \cdots Z_n) \tag{54}$$

Notes: See Sec. 19 for a discussion of this definition.

16 S_n Combinator (Polyadic Elementary Distributor)

Definition:

$$\mathsf{S}_n X Y_1 \cdots Y_n Z \Longrightarrow XZ(Y_1 Z)\cdots(Y_n Z),\ n \geq 0 \tag{55}$$

Reduction to SK:

$$\begin{aligned} \mathsf{S}_0 &= \mathsf{I} && (56)\\ \mathsf{S}_1 &= \mathsf{S} && (57)\\ \mathsf{S}_{n+1} &= \mathsf{BS}_n \circ \mathsf{S} && (58) \end{aligned}$$

Size: $(5n-4)\mathsf{S} + 4(n-1)\mathsf{K} + 9(n-1)\mathsf{A} = 18(n-1)+1$ total for $\mathsf{S}_n,\ n \geq 1$.

Notes: S_n can be replicating or sharing depending on whether S or $\check{\mathsf{S}}$ is used in its recursive definition. If it is sharing, it produces the following structure:

$$\check{\mathsf{S}}_n X Y_1 \cdots Y_n Z \Longrightarrow XZ^{(n)}(Y_1 Z^{(n-1)})\cdots(Y_{n-1}Z^{(1)})(Y_n Z^{(0)}) \tag{59}$$

Let $\hat{\mathsf{S}} = \check{\mathsf{S}}$ or S depending on whether sharing is desired or not.

Equivalences:

$$\begin{aligned} \hat{\mathsf{S}}_{[n]} &= \hat{\mathsf{S}}_n && (60)\\ \hat{\mathsf{S}}_n &= \hat{\Phi}_{n+1}\mathsf{I} && (61)\\ \hat{\mathsf{S}}_n^{[m]} &= \hat{\Phi}_{n+1}^m \mathsf{I} && (62) \end{aligned}$$

The effect of iterating S or $\check{\mathsf{S}}$ is as follows ($m \geq 0$):

$$\begin{aligned} &\mathsf{S}_n^{[m]} X Y_1 \cdots Y_n Z_1 \cdots Z_m \\ &\quad \Longrightarrow \ XZ_1\cdots Z_m(Y_1 Z_1 \cdots Z_m)\cdots(Y_n Z_1 \cdots Z_m) && (63)\\ &\check{\mathsf{S}}_n^{[m]} X Y_1 \cdots Y_n Z_1 \cdots Z_m \\ &\quad \Longrightarrow \ XZ_1^{(n)}\cdots Z_m^{(n)}(Y_1 Z_1^{(n-1)} \cdots Z_m^{(n-1)})\cdots(Y_n Z_1^{(0)} \cdots Z_m^{(0)}) && (64) \end{aligned}$$

17 V Primitive (Sharing Complex)

The sharing primitive (V) is used for constructing non-tree structures, including cyclic structures. It is produced by sharing combinators such as $\check{\mathsf{S}}$, $\check{\mathsf{W}}$, and $\check{\mathsf{Y}}$. Note that a V complex between a combinator and its arguments will block reduction of the combinator, so V complexes appear primarily in structured that are being treated as data.

Primes and parenthesized superscripts on variables are used to indicate informally the sharing of structures. Thus, if there is a single sharing complex above X, then the two links to it will be called X and X'. Notice that both will be "covered" by a sharing complex; if it is necessary to make this explicit, the two links will be written $X^{(0)}$ and X'. If one of these links is replaced by another sharing complex, then the original link and the two new ones will be called X, X', X'', and so forth. Obviously such a notation cannot capture all the possible structures of sharing complexes, but it allows the convenient expression of chains of V complexes, which is the most common case. To go beyond this, diagrams should be used.

18 W Combinator (Elementary Duplicator, Replicating)

Definition:

$$\mathsf{W}XY \Longrightarrow XYY \tag{65}$$

Reduction to SK:

$$\mathsf{W} = \mathsf{CSI} \tag{66}$$
$$\mathsf{W} = \mathsf{S(CI)} \tag{67}$$
$$\mathsf{W} = \mathsf{SS(KI)} \tag{68}$$

Size: $7\mathsf{S} + 6\mathsf{K} + 12\mathsf{A} = 25$ total (Def. 66 or 67).
Equivalences:

$$\mathsf{W}^n FX \Longrightarrow F\overbrace{X\cdots X}^{n+1} \tag{69}$$
$$\mathsf{W}_{(n)}FX_1\cdots X_nY \Longrightarrow FX_1\cdots X_nYY,\ n\geq 0 \tag{70}$$
$$\mathsf{W}_{[n]}XY_1\cdots Y_n \Longrightarrow XY_1Y_1\cdots Y_nY_n,\ n\geq 0 \tag{71}$$

19 $\check{\mathsf{W}}$ Combinator (Elementary Duplicator, Sharing)

Definition:

$$\check{\mathsf{W}}XY \Longrightarrow XY'Y \tag{72}$$

Reduction to SK:

$$\check{\mathsf{W}}_{12} = \mathsf{C}\check{\mathsf{S}}\mathsf{I} \tag{73}$$
$$\check{\mathsf{W}}_{21} = \check{\mathsf{S}}\mathsf{(CI)} \tag{74}$$
$$\check{\mathsf{W}}_{12} = \mathsf{S}\check{\mathsf{S}}\mathsf{(KI)} \tag{75}$$

Notes: $\check{\mathsf{W}}_{12}$ and $\check{\mathsf{W}}_{21}$ are two variants, functionally equivalent to $\check{\mathsf{W}}$, but producing differently ordered links to the sharing (V) complex (see Equivalences below). In the absence of subscripts, we will take $\check{\mathsf{W}}$ to be $\check{\mathsf{W}}_{12}$, since it is a little more convenient to use. Definition 75 is not very useful, because it needlessly begins replication of the first argument of $\check{\mathsf{W}}_{12}$.

Notice that either $\check{\mathsf{W}}$ or $\check{\mathsf{S}}$ may be taken as a primitive sharing operation, since either can be defined in terms of the other. At this time, it looks as though $\check{\mathsf{S}}$ will be the best choice as a primitive, so $\check{\mathsf{W}}$ will be defined by Eq. 73 or 74.

Reaction:

$$U\mathsf{A}_2\check{\mathsf{W}}XY + \mathsf{P}_2\mathsf{VQ} \longrightarrow U\mathsf{A}_2X\mathsf{V}Y + \mathsf{P}\check{\mathsf{W}} + \mathsf{PQ} \tag{76}$$

Reaction Specification:

```
w: Wsh, a: A, a': A, u, x, y, v: V, p: P, p': P, q: Q.
u a, a_1 a', a'_1 w, a'_2 x, a_2 y, p v_1, p' v_2, v q
=> (SharingWreaction)
   u a, a_1 a', a'_1 x, a'_2 v_1, a_2 v_2, v y, p' w, p q.
```

Equivalences:

$$\check{\mathsf{W}}_{12}XY \implies XY'Y \tag{77}$$

$$\check{\mathsf{W}}_{21}XY \implies XYY' \tag{78}$$

$$\check{\mathsf{W}}_{12}^{n}XY \implies X\underbrace{Y^{(n)}\cdots Y^{(2)}Y^{(1)}Y^{(0)}}_{n+1} \tag{79}$$

$$\check{\mathsf{W}}_{[n]}XY_1\cdots Y_n \implies XY_1'Y_1\cdots Y_n'Y_n,\ n\geq 0 \tag{80}$$

Notes: The superscripts on Y in Eq. 79 represent successive sharings of Y (see Sec. 17).

20 W_* Combinator (Pure Duplicator)

Definition:

$$\mathsf{W}_*X \implies XX \tag{81}$$

Reduction to SK:

$$\mathsf{W}_* = \mathsf{WI} \tag{82}$$

Size: $8\mathsf{S} + 8\mathsf{K} + 15\mathsf{A} = 31$ total.

21 Y Combinator (Elementary Fixed-point, Replicating)

Definition:

$$\mathsf{Y}F \implies X(\mathsf{Y}X) \tag{83}$$

Reduction to SK:

$$\mathsf{Y} = \mathsf{SSK(S(K(SS(S(SSK))))K)} \tag{84}$$

Size: $8\mathsf{S} + 4\mathsf{K} + 11\mathsf{A} = 23$ total.

Equivalences:

$$\begin{aligned} \mathsf{Y} &= \mathsf{WS(BWB)} && (85)\\ \mathsf{Y} &= \mathsf{SSI(SB(K(SII)))} && (86)\\ \mathsf{Y} &= ZZ \text{ where } Z = \mathsf{W(B(SI))} && (87)\\ \mathsf{Y} &= \mathsf{WI} \circ \mathsf{W} \circ \mathsf{B} && (88) \end{aligned}$$

Notes: Definition 84 by John Tromp [2] may be the shortest definition in terms of SK (12 combinators). Definitions by Curry and Turing are longer (18 and 20, respectively).

22 $\check{\mathsf{Y}}$ Combinator (Elementary Fixed-point, Sharing)

Definition:

$$\check{\mathsf{Y}}X \Longrightarrow y^{(1)} \quad \text{where } y \equiv Fy^{(0)} \tag{89}$$

Reaction:

$$U\mathsf{A}\check{\mathsf{Y}}X + \mathsf{P}_2\mathsf{VQ} \longrightarrow U\mathsf{VA}X + \mathsf{P}\check{\mathsf{Y}} + \mathsf{PQ} \tag{90}$$

Reaction Specification:

```
y: Ysh, a: A, v: V, x, p: P, p': P, q: Q.
u a, a_1 y, a_2 x, p v_1, p' v_2, v q
=> (SharingYreaction)
   u v_1, v a, a_1 x, a_2 v_2, p' y, p q.
```

Notes: The following illustrates the self-sharing cycle created by $\check{\mathsf{Y}}F$:

$$\begin{aligned} \check{\mathsf{Y}}F &= y\\ &= Fy'\\ &= F(Fy')'\\ &= F(F'y'')\\ &= F(F'(Fy')'')\\ &= F(F'(F''y'''))\\ &\vdots\\ &= F(F'(F''(F'''(F^{(4)}(F^{(5)}\cdots))))) \end{aligned}$$

Of course, it is the A complex that is shared, not F, as the notation suggests.

23 Z Combinators (Iterators or Church Numerals)

Definition:

$$\mathsf{Z}_n X = X^n$$

Reduction to SK:

$$\begin{aligned} \mathsf{Z}_0 &= \mathsf{KI} && (91) \\ \mathsf{Z}_{n+1} &= \mathsf{SBZ}_n && (92) \end{aligned}$$

Size: $(3n+1)\mathsf{S} + (2n+3)\mathsf{K} + (5n+3)\mathsf{A} = 10n + 7$ total, for Z_n.
Equivalences:

$$\begin{aligned} \mathsf{Z}_{m+n} &= \Phi\mathsf{BZ}_m\mathsf{Z}_n && (93) \\ \mathsf{Z}_{mn} &= \mathsf{Z}_m \circ \mathsf{Z}_n && (94) \\ \mathsf{Z}_{n^m} &= \mathsf{Z}_m\mathsf{Z}_n && (95) \end{aligned}$$

24 Φ Combinator (Dyadic Compositor)

Definition:

$$\Phi XYZU \Longrightarrow X(YU)(ZU) \tag{96}$$

Reduction to SK:

$$\Phi = \mathsf{B(BS)B} \tag{97}$$

Size: $7\mathsf{S} + 6\mathsf{K} + 12\mathsf{A} = 25$ total (Def. 97).
Equivalences:

$$\Phi^n FGHX_1 \cdots X_n \Longrightarrow F(GX_1 \cdots X_n)(HX_1 \cdots X_n) \tag{98}$$

Notes: Φ composes an order-2 combinator with two order-1 combinators to produce an order-1 combinator.

25 Φ_n Combinator (Polyadic Compositor)

Definition:

$$\Phi_n XY_1 \cdots Y_n Z \Longrightarrow X(Y_1 Z) \cdots (Y_n Z), \ n \geq 0 \tag{99}$$

Reduction to SK:

$$\Phi_n = \mathsf{S}_n \circ \mathsf{K} \tag{100}$$

Size: $(5n-2)\mathsf{S} + (4n-1)\mathsf{K} + (9n-4)\mathsf{A} = 18n - 7$ total for $\Phi_n, \ n \geq 1$.

Notes: Φ_n composes an order-n combinator with n order-1 combinators to produce an order-1 combinator.

Φ_n can be replicating or sharing ($\check{\Phi}_n$), depending on whether S_n or $\check{\mathsf{S}}_n$ is used in definition 100. If it is sharing, then the following structure is generated:

$$\check{\Phi}_n X Y_1 \cdots Y_n Z \Longrightarrow X(Y_1 Z^{(n-1)}) \cdots (Y_{n-1} Z^{(1)})(Y_n Z^{(0)}) \tag{101}$$

Equivalences:

$$\Phi_{n+1} \quad = \quad \mathsf{BS}_n \circ \mathsf{B} \tag{102}$$

The effect of iterating Φ or $\check{\Phi}$ is as follows ($m \geq 0$):

$$\begin{aligned} & \Phi_n^m X Y_1 \cdots Y_n Z_1 \cdots Z_m \\ & \quad \Longrightarrow \quad X(Y_1 Z_1 \cdots Z_m) \cdots (Y_n Z_1 \cdots Z_m) \end{aligned} \tag{103}$$

$$\begin{aligned} & \check{\Phi}_n^m X Y_1 \cdots Y_n Z_1 \cdots Z_m \\ & \quad \Longrightarrow \quad X(Y_1 Z_1^{(n-1)} \cdots Z_m^{(n-1)}) \cdots (Y_n Z_1^{(0)} \cdots Z_m^{(0)}) \end{aligned} \tag{104}$$

$$\begin{aligned} & \Phi_{n+1}^m \mathsf{I} X Y_1 \cdots Y_n Z_1 \cdots Z_m \\ & \quad \Longrightarrow \quad X Z_1 \cdots Z_m (Y_1 Z_1 \cdots Z_m) \cdots (Y_n Z_1 \cdots Z_m) \end{aligned} \tag{105}$$

26 X_n Combinator (Chi Distributor)

Definition:

$$\mathrm{X}_n F G U_1 \cdots U_n \Longrightarrow F(GU_1) \cdots (GU_n),\ n \geq 0 \tag{106}$$

Reduction to SK**:**

$$\mathrm{X}_0 \quad = \quad \mathsf{K} \tag{107}$$

$$\mathrm{X}_1 \quad = \quad \mathsf{B} \tag{108}$$

$$\mathrm{X}_{n+1} \quad = \quad \mathsf{W}_{(1)}(\mathsf{C}_{(2)}(\mathsf{B}^3 \mathrm{X}_n \mathsf{B})) \tag{109}$$

Size: $30n\mathsf{S} + (28n+1)\mathsf{K} + 58n\mathsf{A} = 116n + 1$ total, for X_n, $n \geq 0$.
Equivalences:

$$\mathrm{X}_n \quad = \quad (\mathsf{W}_{(1)} \circ \mathsf{C}_{(2)} \circ \mathsf{CB}^3\mathsf{B})^n \mathsf{K} \tag{110}$$

$$\mathrm{X}_{n+1} \quad = \quad \mathsf{W} \circ \mathsf{C}^{[n+1]} \circ \mathsf{B}^{n+1}\mathsf{B} \circ \mathrm{X}_n \tag{111}$$

$$\mathrm{X}_n \quad = \quad (\mathsf{W} \circ \mathsf{C}^{[n+1]} \circ \mathsf{B}^{n+1}\mathsf{B})^n \circ \mathsf{K} \tag{112}$$

$$\mathrm{X}_2 \quad = \quad \Psi \tag{113}$$

Notes: X_n composes an order-n combinator with an order-1 combinator, used n times, to produce an order-n combinator.

The effect of a left reduction is:

$$\begin{aligned} &(\mathrm{X}_n)_{[m]} F G_1 \cdots G_m U_1 \cdots U_n \\ &\quad \Longrightarrow \quad F(G_1(\cdots(G_m U_1)\cdots))\cdots(G_1(\cdots(G_m U_n)\cdots)) \end{aligned} \tag{114}$$

If $\check{\mathsf{W}}_{21}$ is used in Definition 109, then

$$\check{\mathrm{X}}_n F G U_1 \cdots U_n \Longrightarrow F(G^{(0)} U_1) \cdots (G^{(n-1)} U_n) \tag{115}$$

27 Ψ Combinator (Ψ Distributor)

Definition:

$$\Psi XYUV \Longrightarrow X(YU)(YV) \tag{116}$$

Reduction to SK:

$$\Psi = \Phi(\Phi(\Phi\mathsf{B}))\mathsf{B}(\mathsf{KK}) \tag{117}$$

Size: $26\mathsf{S} + 24\mathsf{K} + 49\mathsf{A} = 99$ total.
Equivalences:

$$\begin{aligned} \Psi &= \mathsf{B}(\mathsf{BW}(\mathsf{BC}))(\mathsf{BB}(\mathsf{BB})) && (118) \\ \Psi &= \Phi^4\mathsf{BB}(\mathsf{KK}) && (119) \\ \Psi &= \mathsf{C}_{[2]}\Phi^3(\mathsf{BK})\mathsf{K} && (120) \\ \Psi &= \mathsf{W} \circ \mathsf{C}_{(1)} \circ \mathsf{B} \circ \mathsf{B}_{(1)} && (121) \\ \Psi &= \mathsf{S}(\mathsf{B}(\mathsf{BS}(\mathsf{B}(\mathsf{BS}(\mathsf{BB}))))\mathsf{B})(\mathsf{KK}) && (122) \end{aligned}$$

Notes: Ψ composes an order-2 combinator with an order-1 combinator, used twice, to yield an order-2 combinator.

Other Notation

28 Composition

Definition:

$$X \circ Y = \mathsf{B}XY \tag{123}$$

Size: $2\mathsf{S} + 2\mathsf{K} + 5\mathsf{A} = 9$ total, plus $|X| + |Y|$.
Equivalences:

$$\begin{aligned} X \circ \mathsf{I} &= \mathsf{I} \circ X = X & (124)\\ X \circ (Y \circ Z) &= (X \circ Y) \circ Z & (125)\\ \mathsf{B}(X \circ Y) &= \mathsf{B}X \circ \mathsf{B}Y & (126) \end{aligned}$$

29 Powers

Definition:

$$(X \circ Y)Z \Longrightarrow X(YZ) \tag{127}$$

Reduction to SK:

$$\begin{aligned} X^0 &= \mathsf{I} & (128)\\ X^1 &= X & (129)\\ X^{n+1} &= X \circ X^n & (130) \end{aligned}$$

Size: $2(n-1)\mathsf{S} + 2(n-1)\mathsf{K} + 5(n-1)\mathsf{A} = 9(n-1)$ total, plus $n|X|$, for X^n, $n \geq 1$.
Equivalences:

$$\begin{aligned} X^m \circ X^n &= X^{m+n} & (131)\\ (X^m)^n &= X^{mn} & (132)\\ (\mathsf{B}X)^m &= \mathsf{B}(X^m) & (133) \end{aligned}$$

30 Deferred Combinators

Definition:

$$\begin{aligned} X_{(0)} &= X & (134)\\ X_{(n+1)} &= \mathsf{B}X_{(n)} & (135)\\ & & (136) \end{aligned}$$

Size: $2n\mathsf{S} + 2n\mathsf{K} + 4n\mathsf{A} = 8n$ total, plus $|X|$, for $X_{(n)}$, $n \geq 0$.
Equivalences:

$$\begin{aligned} F_{(n)}X_0X_1\cdots X_n &\implies F(X_0X_1\cdots X_n) && (137)\\ X_{(m+n)} &= \mathsf{B}^m X_{(n)} && (138) \end{aligned}$$

Notes: If F is regular (p. 2), $FGY_1\cdots Y_n \Longrightarrow GZ_1\cdots Z_m$, then

$$F_{(k)}GX_1\cdots X_kY_1\cdots Y_n \Longrightarrow GX_1\cdots X_kZ_1\cdots Z_m. \qquad (139)$$

That is, $F_{(k)}$ *defers* the action of F by k steps. Since B, C, I, K, and W are regular, we have Eqs. 7, 18, 36, 41, and 70.

31 Left Reduction

Definition:

$$\begin{aligned} X_{[0]} &= \mathsf{I} && (140)\\ X_{[1]} &= X && (141)\\ X_{[n+1]} &= \mathsf{B}X_{[n]} \circ X && (142) \end{aligned}$$

Size: $4(n-1)\mathsf{S} + 4(n-1)\mathsf{K} + 9(n-1)\mathsf{A} = 17(n-1)$ total, plus $n|X|$, for $X_{[n]}$, $n \geq 1$.
Equivalences:

$$\begin{aligned} F_{[n]}X_0X_1\cdots X_n &\implies F(F\cdots(F(FX_0X_1)X_2)\cdots X_{n-1})X_n && (143)\\ F_{[n+1]}X_0X_1\cdots X_n &\implies F_{[n]}(FX_0X_1)X_2\cdots X_n && (144)\\ X_{[n+1]} &= \mathsf{B}^nX \circ \mathsf{B}^{n-1}X \circ \cdots \circ \mathsf{B}^2X \circ \mathsf{B}X \circ X && (145)\\ X_{[n+1]} &= X_{(n)} \circ X_{(n-1)} \circ \cdots \circ X_{(2)} \circ X_{(1)} \circ X_{(0)} && (146)\\ X_{[n+1]} &= (\mathsf{CB}^2X)^nX && (147)\\ X_{[m+n]} &= \mathsf{B}^mX_{[n]} \circ X_{[m]} && (148) \end{aligned}$$

Notes: $F_{[n]}$ can be called a *left reduction* [3]. To see this, write F in infix form, $Fxy = x \diamond y$ and assume $\diamond$ associates to the left (so $x \diamond y \diamond z = (x \diamond y) \diamond z$). Then:

$$F_{[n]}x_0x_1\cdots x_n = x_0 \diamond x_1 \diamond \cdots \diamond x_n.$$

If F is an order-2 combinator, then $F_{[n]}$ is a combinator of order $n+1$.

For F regular,

$$F_{[n]} = (\mathsf{CB}^2F)\mathsf{I} \qquad (149)$$

32 Polyadic Extension

Definition:

$$X^{[0]} = \mathsf{I} \tag{150}$$
$$X^{[1]} = X \tag{151}$$
$$X^{[n+1]} = X \circ \mathsf{B}X^{[n]} \tag{152}$$

Size: $4(n-1)\mathsf{S} + 4(n-1)\mathsf{K} + 9(n-1)\mathsf{A} = 17(n-1)$ total, plus $n|X|$, for $X^{[n]}$, $n \geq 1$.
Equivalences:

$$X^{[n+1]} = (\mathsf{B}^2 X \mathsf{B})^n X \tag{153}$$

Notes: If F is regular,

$$F^{[n]} = (\mathsf{B}^2 X \mathsf{B})^n \mathsf{I} \tag{154}$$
$$F^{[n+1]} = F \circ \mathsf{B}F \circ \cdots \circ \mathsf{B}^n F \tag{155}$$
$$F^{[n+1]} = F_{(0)} \circ F_{(1)} \circ \cdots \circ F_{(n)} \tag{156}$$
$$F^{[m+n]} = F^{[m]} \circ \mathsf{B}^m F^{[n]} \tag{157}$$

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[3] Bruce J. MacLennan. *Functional Programming: Practice and Theory*. Addison-Wesley, Reading, 1990.