Is $n^2 + n + 1 = O(n^2)$? To prove this, we need to find a constant c such that $cn^2 \geq n^2 + n + 1$. Let $c = 2$ – that should work. Now we need to find a constant x such that for all $n >= x$, $2n^2 \geq n^2 + n + 1$. We'll try $x = 10$.

Let's proceed by an inductive argument. To make our life simpler, let $f(n) = 2n^2$, and $g(n) = n^2 + n + 1$. When n = 10, $f(n) = 200$ and $g(n) = 111$, so $f(x) > g(x)$. Now, let's assume that our statement is true for all values between 10 and n for some n. We already know that this is true for $n = 10$. Let's look at $n + 1$:

$$\begin{aligned} f(n+1) &= 2(n+1)^2 \\ &= 2n^2 + 4n + 2 \\ &= f(n) + 4n + 2 \end{aligned}$$

$$\begin{aligned} g(n+1) &= (n+1)^2 + (n+1) + 1 \\ &= n^2 + 2n + 1 + n + 1 + 1 \\ &= n^2 + 3n + 3 \\ &= (n^2 + n + 1) + 2n + 2 \\ &= g(n) + 2n + 2 \end{aligned}$$

From our inductive hypothesis, we know $f(n) \geq g(n)$, thus:

$$f(n) + 4n + 2 \geq g(n) + 4n + 2$$

Since $n \geq 10$, $4n + 2 > 2n + 2$, and therefore:

$$\begin{aligned} f(n) + 4n + 2 &> g(n) + 2n + 2 \\ f(n+1) &> g(n+1) \end{aligned}$$

Therefore, for all $n >= 10$, $2n^2 > n^2+n+1$, meaning $2n^2 \geq n^2+n+1$, and therefore $n^2+n+1 = O(n^2)$ ■.